A Topological Reconstruction Theorem for $\mathcal{D}^\infty$-Modules

by

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Abstract

In this paper, we prove that any perfect complex of $\mathcal{D}^\infty$-modules may be reconstructed from its holomorphic solution complex provided that we keep track of the natural topology of this last complex. This is to be compared with the reconstruction theorem for regular holonomic $\mathcal{D}$-modules which follows from the well-known Riemann-Hilbert correspondence. To obtain our result, we consider sheaves of holomorphic functions as sheaves with values in the category of ind-Banach spaces and study some of their homological properties. In particular, we prove that a Künneth formula holds for them and we compute their Poincaré-Verdier duals. As a corollary, we obtain the form of the kernels of "continuous" cohomological correspondences between sheaves of holomorphic forms. This allows us to prove a kind of holomorphic Schwartz' kernel theorem and to show that $\mathcal{D}^\infty \simeq \mathrm{R}\mathcal{H}omtop(\mathcal{O}, \mathcal{O})$. Our reconstruction theorem is a direct consequence of this last isomorphism. Note that the main problem is the vanishing of the topological Ext's and that this vanishing is a consequence of the acyclicity theorems for DFN spaces which are established in the paper.

Contents

0 Introduction

In algebraic analysis, one represents the systems of analytic linear partial differential equations on a complex analytic manifold X by modules over the ring $\mathcal{D}_X$ of linear partial differential operators with analytic coefficients. Using this representation, the holomorphic solutions of the homogeneous system associated to the $\mathcal{D}_X$-module $\mathcal{M}$ correspond to

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$$

where $\mathcal{O}_X$ denotes the $\mathcal{D}_X$-module of holomorphic functions. If one wants also to take into consideration the compatibility conditions, one has to study the full solution complex

$$\mathcal{S}ol(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$$

in the derived category $D^+(\mathbb{C}_X)$ of sheaves of $\mathbb{C}$-vector spaces. In [4] (see also [7]), it was shown that the functor $\mathcal{S}ol$ induces an equivalence between the derived category formed by the complexes of regular holonomic $\mathcal{D}_X$-modules and that formed by the complexes of $\mathbb{C}$-constructible $\mathbb{C}_X$-modules. This equivalence is usually called the Riemann-Hilbert correspondence. One of its corollaries is that it is possible to reconstruct a complex of regular holonomic $\mathcal{D}_X$-modules from its complex of holomorphic solutions.

Our aim in this paper is to extend this reconstruction theorem to perfect complexes of $\mathcal{D}_X^\infty$-modules by taking into account the natural topology of the complex of holomorphic solutions. Informally, the relation we will obtain is of the type

$$\mathcal{M} \simeq \mathrm{R}\mathcal{H}omtop(\mathcal{S}ol(\mathcal{M}), \mathcal{O}_X)$$

and will follow from the fact that

$$\mathcal{D}_X^\infty \simeq \mathrm{R}\mathcal{H}omtop(\mathcal{O}_X, \mathcal{O}_X).$$

To give a meaning to these formulas, we will have to work in the derived category of sheaves with values in the category of ind-objects of the category of Banach spaces using the techniques and results of [16].

Let us now describe with some details the content of this paper.

Following [13], we denote $\mathcal{B}an$ (resp. $\mathcal{F}r$, $\mathcal{T}c$) the quasi-abelian category of Banach spaces (resp. Fréchet spaces, arbitrary locally convex topological vector spaces). Let us recall (see e.g. [10]) that, for any set I, the space $l^1(I)$ (resp. $l^\infty(I)$) of summable (resp. bounded) sequences of $\mathbb{C}$ indexed by I is projective (resp. injective) in $\mathcal{B}an$. Using these spaces, one shows easily that $\mathcal{B}an$ has enough injective and projective objects. Recall also that the category $\mathcal{B}an$ has a canonical structure of closed additive category given by a right exact tensor product

$$\hat{\otimes} : \mathcal{B}an \times \mathcal{B}an \longrightarrow \mathcal{B}an$$

and a left exact internal Hom

$$\mathrm{L} : \mathcal{B}an^{\mathrm{op}} \times \mathcal{B}an \longrightarrow \mathcal{B}an.$$

Denoting $\hat{\otimes}^L$ the left derived functor of $\hat{\otimes}$ and RL the right derived functor of L, we have the adjunction formula

$$\mathrm{RHom}\,(E \,\hat{\otimes}^L\, F, G) \simeq \mathrm{RHom}\,(E, \mathrm{RL}\,(F, G)).$$

Therefore, using [16], we see that the category $\mathcal{I}nd(\mathcal{B}an)$ of ind-objects of $\mathcal{B}an$ is an elementary closed quasi-abelian category. It follows that sheaves with values in $\mathcal{I}nd(\mathcal{B}an)$ share most of the usual properties of abelian sheaves (including Künneth Theorem and Poincaré-Verdier duality). Recall that, in $\mathcal{I}nd(\mathcal{B}an)$, the internal tensor product

$$\hat{\otimes} : \mathcal{I}nd(\mathcal{B}an) \times \mathcal{I}nd(\mathcal{B}an) \longrightarrow \mathcal{I}nd(\mathcal{B}an)$$

and the internal Hom functor

$$\mathrm{L} : (\mathcal{I}nd(\mathcal{B}an))^{\mathrm{op}} \times \mathcal{I}nd(\mathcal{B}an) \longrightarrow \mathcal{I}nd(\mathcal{B}an)$$

are characterized by

$$\left(\text{``}\varinjlim_{i \in I}\text{''}\, E_i \right) \hat{\otimes} \left( \text{``}\varinjlim_{j \in J}\text{''}\, F_j \right) = \varinjlim_{i \in I} \varinjlim_{j \in J} \text{``}E_i \,\hat{\otimes}\, F_j\text{''}$$

and

$$\mathrm{L}\left(\text{``}\varinjlim_{i \in I}\text{''}\, E_i, \text{``}\varinjlim_{j \in J}\text{''}\, F_j \right) = \varprojlim_{i \in I} \varinjlim_{j \in J} \text{``}\mathrm{L}\,(E_i, F_j)\text{''}.$$

To fix the notations, recall also that the internal tensor product (resp. internal Hom functor) for sheaves with values in $\mathcal{I}nd(\mathcal{B}an)$ are denoted $\hat{\otimes}$ (resp. $\mathcal{L}$).

In the first section of this paper, we study the functor $\mathrm{IB} : \mathcal{T}c \to \mathcal{I}nd(\mathcal{B}an)$ defined by setting

$$\mathrm{IB}(E) = \text{``}\varinjlim_{B \in \mathcal{B}_E}\text{''}\, \widehat{E}_B$$

where $\mathcal{B}_E$ is the set of absolutely convex bounded subsets of E and E_B the linear hull of B. We establish the properties of this functor we need in the rest of the paper. More precisely, we prove that if E is bornological and F complete, then

$$\mathrm{Hom}_{\mathcal{I}nd(\mathcal{B}an)}(\mathrm{IB}(E), \mathrm{IB}(F)) \simeq \mathrm{Hom}_{\mathcal{T}c}(E, F)$$

and

$$\mathrm{IB}(\mathrm{L}_b(E, F)) \simeq \mathrm{L}(\mathrm{IB}(E), \mathrm{IB}(F))$$

where $\mathrm{L}_b(E, F)$ is the vector space $\mathrm{Hom}_{\mathcal{T}c}(E, F)$ endowed with the system of semi-norms

$$\{p_B : p \text{ continuous semi-norm of } F, B \text{ bounded subset of } E\}$$

where

$$p_B(h) = \sup_{e \in B} p(h(e)).$$

Moreover, we show that IB is compatible with projective limits of filtering projective systems of complete spaces. We show also its compatibility with complete inductive limits of injective inductive systems of Fréchet spaces indexed by $\mathbb{N}$.

The second section is devoted to the proof of some acyclicity results for L and $\hat{\otimes}$ in $\mathcal{I}nd(\mathcal{B}an)$. First, we show that if E is a DFN space and if F is a Fréchet space, then both $LH^k(\mathrm{RHom}(\mathrm{IB}(E), \mathrm{IB}(F)))$ and $LH^k(\mathrm{RL}(\mathrm{IB}(E), \mathrm{IB}(F)))$ are 0 for $k \neq 0$. (Note that a similar result was obtained for the category $\mathcal{T}c$ by Palamodov in [8].) Next, we establish that if E and F are objects of $\mathcal{I}nd(\mathcal{B}an)$ with E nuclear, then

$$E \hat{\otimes}^L F \simeq E \hat{\otimes} F. \tag{*}$$

We start Section 3 by proving that if X is a topological space with a countable basis and if F is a presheaf of Fréchet spaces on X which is a sheaf of vector spaces, then

$$U \mapsto \mathrm{IB}(F(U)) \qquad (U \text{ open of } X)$$

is a sheaf with values in $\mathcal{I}nd(\mathcal{B}an)$. This shows, in particular, that $\mathrm{IB}(\mathcal{O}_X)$ is a sheaf with values in $\mathcal{I}nd(\mathcal{B}an)$ for any complex analytic manifold X. We end the section by establishing that

$$\mathrm{R}\Gamma(U, \mathrm{IB}(\mathcal{O}_X)) \simeq \Gamma(U, \mathrm{IB}(\mathcal{O}_X))$$

if U is an open subset of X such that $H^k(U, \mathcal{O}_X) \simeq 0$ ($k \neq 0$). This result may be viewed as a topological version of Cartan's Theorem B. As a corollary, if X is a Stein manifold, we get a similar isomorphism with U replaced by any holomorphically convex compact subset of X.

In Section 4, using (*), we show that

$$\mathrm{IB}(\mathcal{O}_X) \hat{\boxtimes}^L \mathrm{IB}(\mathcal{O}_Y) \simeq \mathrm{IB}(\mathcal{O}_{X \times Y})$$

for any complex analytic manifolds X and Y. This allows us to obtain a topological Künneth Theorem for holomorphic cohomology.

Section 5 is devoted to the proof that, for any complex analytic manifold X of dimension d_X, the Poincaré dual of $\mathrm{IB}(\mathcal{O}_X)$ is isomorphic to $\mathrm{IB}(\Omega_X)[d_X]$. Since the problem is of local nature, we find, by a series of reductions using the results established in the previous sections, that it is sufficient to show that, if P is a closed interval of $\mathbb{C}$ and V is an open interval of $\mathbb{C}^n$, then

$$\mathrm{R}\Gamma_{P \times V}(\mathbb{C} \times V, \mathrm{IB}(\mathcal{O}_{\mathbb{C} \times V})) \simeq \mathrm{L}\left(\mathrm{IB}(\mathcal{O}_{\mathbb{C}}(P)), \mathrm{IB}(\mathcal{O}_V(V))\right)[-1].$$

This isomorphism is obtained by proving that, in this situation, one has a split exact sequence of the form

$$0 \to \mathcal{O}_{\mathbb{C} \times V}(\mathbb{C} \times V) \to \mathcal{O}_{\mathbb{C} \times V}((\mathbb{C} \setminus P) \times V) \to \mathrm{L}_b(\mathcal{O}_{\mathbb{C}}(P), \mathcal{O}_V(V)) \to 0$$

in $\mathcal{T}c$.

We begin Section 6 by giving the general form of the kernels of continuous cohomological correspondences between sheaves of holomorphic differential forms. More precisely, we show that, if X, Y are complex analytic manifolds of dimension d_X, d_Y, then

$$\mathrm{IB}(\Omega_{X \times Y}^{(d_X - r, s)})[d_X] \simeq \mathrm{R}\mathcal{L}\left(q_X^{-1}\,\mathrm{IB}(\Omega_X^r), q_Y^!\,\mathrm{IB}(\Omega_Y^s)\right).$$

As a consequence, we get that for any morphism of complex analytic manifolds $f : X \to Y$, we have a canonical isomorphism

$$\mathrm{R}\mathcal{L}\left(f^{-1}\,\mathrm{IB}(\mathcal{O}_Y), \mathrm{IB}(\mathcal{O}_X)\right) \simeq \delta_f^{-1}\,\mathrm{R}\Gamma_{\Delta_f}\,\mathrm{IB}(\Omega_{X \times Y}^{(0, d_Y)})[d_Y]$$

where Δ_f is the graph of f in $X \times Y$ and $\delta_f : X \to X \times Y$ is the associated graph embedding. In particular,

$$LH^k(\mathrm{R}\mathcal{L}\left(f^{-1}\,\mathrm{IB}(\mathcal{O}_Y), \mathrm{IB}(\mathcal{O}_X)\right)) = 0$$

for $k \neq 0$ and

$$R\mathcal{H}om\left(f^{-1}\,\mathrm{IB}(\mathcal{O}_Y), \mathrm{IB}(\mathcal{O}_X)\right) \simeq \mathcal{D}_{X \to Y}^\infty. \tag{**}$$

Note that this contains the fact that continuous endomorphisms of $\mathcal{O}_X$ may be identified with partial differential operators of infinite order as was conjectured by Sato and established by Ishimura in [3].

We start the last section by proving an abstract reconstruction theorem for perfect complexes of modules over a ring in the closed category $\mathcal{S}hv(X; \mathcal{I}nd(\mathcal{B}an))$. Thanks to the embedding functor

$$\tilde{I}_{\mathcal{V}} : \mathcal{S}hv(X; \mathcal{V}) \to \mathcal{S}hv(X; \mathcal{I}nd(\mathcal{B}an))$$

(where $\mathcal{V}$ denotes the category of $\mathbb{C}$-vector spaces) we are also able to prove a similar formula for perfect complexes of modules over an ordinary sheaf of rings. Using (**) with $f = \mathrm{id}_X$, we get a topological reconstruction theorem for $\mathcal{D}_X^\infty$-modules. More precisely, we prove that the functors

$$\mathrm{R}\mathcal{L}_{\tilde{I}_\mathcal{V}(\mathcal{D}_X^\infty)}(\tilde{I}_\mathcal{V}(\cdot), \mathrm{IB}(\mathcal{O}_X)) : D^-(\mathcal{M}od(\mathcal{D}_X^\infty)) \to D^+(\mathcal{S}hv(X; \mathcal{I}nd(\mathcal{B}an)))$$

and

$$R\mathcal{H}om\,(\cdot, \mathrm{IB}(\mathcal{O}_X)) : D^-(\mathcal{S}hv(X; \mathcal{I}nd(\mathcal{B}an))) \to D^+(\mathcal{M}od(\mathcal{D}_X^\infty))$$

are well-defined and that

$$R\mathcal{H}om\,(\mathrm{R}\mathcal{L}_{\tilde{I}_\mathcal{V}(\mathcal{D}_X^\infty)}(\tilde{I}_\mathcal{V}(\mathcal{M}), \mathrm{IB}(\mathcal{O}_X)), \mathrm{IB}(\mathcal{O}_X)) \simeq \mathcal{M}$$

for any perfect complex of $\mathcal{D}_X^\infty$-modules $\mathcal{M}$. Note that the image of $\mathcal{M}$ by the first functor above is a kind of topologized version of the holomorphic solution complex of $\mathcal{M}$ and that the preceding formula may be viewed as a way to reconstruct a perfect system of analytic partial differential equations of infinite order from its holomorphic solutions.

1   The functor $\mathrm{IB} : \mathcal{T}c \to \mathcal{I}nd(\mathcal{B}an)$

For any object E of $\mathcal{T}c$, we denote by $\mathcal{B}_E$ the set of absolutely convex bounded subsets of E and by $\overline{\mathcal{B}}_E$ the set of closed absolutely convex bounded subsets of E. If $B \in \mathcal{B}_E$, we denote E_B the linear hull of B and p_B the gauge semi-norm of E_B associated to B.

Definition 1.1. To define the functor

$$\mathrm{IB} : \mathcal{T}c \to \mathcal{I}nd(\mathcal{B}an)$$

we proceed as follows. For any object E of $\mathcal{T}c$, we set

$$\mathrm{IB}(E) = \text{``}\varinjlim_{B \in \mathcal{B}_E}\text{''}\,\widehat{E}_B.$$

Consider a morphism $f : E \to F$ of $\mathcal{T}c$. For any $B \in \mathcal{B}_E$, $f(B) \in \mathcal{B}_F$. Hence, f induces a morphism $\widehat{E}_B \to \widehat{F}_{f(B)}$. This morphism being functorial in B, we obtain a morphism

$$\text{``}\varinjlim_{B \in \mathcal{B}_E}\text{''}\,\widehat{E}_B \longrightarrow \text{``}\varinjlim_{B \in \mathcal{B}_E}\text{''}\,\widehat{F}_{f(B)}$$

in $\mathcal{I}nd(\mathcal{B}an)$. We define

$$\mathrm{IB}(f) : \mathrm{IB}(E) \to \mathrm{IB}(F)$$

by composing the preceding morphism with the canonical morphism

$$\text{``}\varinjlim_{B\in\mathcal{B}_E}\text{''}\,\widehat{F}_{f(B)} \longrightarrow \text{``}\varinjlim_{B\in\mathcal{B}_F}\text{''}\,\widehat{F}_B.$$

Remark 1.2. If E is a Banach space, then

$$\text{IB}(E) \simeq \text{``}E\text{''}.$$

As a matter of fact, since any bounded subset of E is included in a ball $b(\rho)$ centered at the origin, we have

$$\text{IB}(E) \simeq \text{``}\varinjlim_{\rho>0}\text{''}\,E_{b(\rho)}$$

and the conclusion follows from the isomorphism $E_{b(\rho)} \simeq E$.

Lemma 1.3. *Let E and F be two objects of $\mathcal{T}c$. Then,*

$$\varprojlim_{B\in\mathcal{B}_E}\varinjlim_{B'\in\mathcal{B}_F}\text{Hom}_{\mathcal{T}c}(E_B,F_{B'}) \simeq B(E,F)$$

where

$$B(E,F) = \{f : E \longrightarrow F : f \text{ linear}, f(B) \text{ bounded in } F \text{ if } B \text{ bounded in } E\}.$$

Remark 1.4. If E and F are objects of $\mathcal{T}c$, we have

$$\text{Hom}_{\mathcal{T}c}(E,F) \subset B(E,F).$$

In general, this inclusion is strict but, as is well-known, it turns into an equality if E is bornological (i.e. if any absolutely convex subset of E that absorbs any bounded subset is a neighborhood of zero).

Proposition 1.5. *Let E and F be two objects of $\mathcal{T}c$. If E is bornological and F complete, then*

$$\text{Hom}_{\mathcal{I}nd(\mathcal{B}an)}(\text{IB}(E),\text{IB}(F)) \simeq \text{Hom}_{\mathcal{T}c}(E,F).$$

Proof. Since the inclusion $\overline{\mathcal{B}}_F \subset \mathcal{B}_F$ is cofinal, we have

$$\text{Hom}_{\mathcal{I}nd(\mathcal{B}an)}(\text{IB}(E),\text{IB}(F)) \simeq \text{Hom}_{\mathcal{I}nd(\mathcal{B}an)}\left(\text{``}\varinjlim_{B\in\mathcal{B}_E}\text{''}\,\widehat{E}_B, \text{``}\varinjlim_{B'\in\overline{\mathcal{B}}_F}\text{''}\,\widehat{F}_{B'}\right)$$

$$\simeq \varprojlim_{B\in\mathcal{B}_E}\varinjlim_{B'\in\overline{\mathcal{B}}_F}\text{Hom}_{\mathcal{B}an}(\widehat{E}_B,\widehat{F}_{B'}).$$

Since F is complete, $F_{B'}$ is a Banach space and

$$\text{Hom}_{\mathcal{B}an}(\widehat{E}_B,\widehat{F}_{B'}) \simeq \text{Hom}_{\mathcal{T}c}(E_B,F_{B'}).$$

It follows that

$$\mathrm{Hom}_{\mathcal{I}nd(\mathcal{B}an)}(\mathrm{IB}(E), \mathrm{IB}(F)) \simeq \varprojlim_{B \in \mathcal{B}_E} \varinjlim_{B' \in \mathcal{B}_F} \mathrm{Hom}_{\mathcal{T}c}(E_B, F_{B'})$$

$$\simeq B(E, F) \simeq \mathrm{Hom}_{\mathcal{T}c}(E, F)$$

where the second isomorphism follows from Lemma 1.3 and the last isomorphism from Remark 1.4. $\qquad\square$

Proposition 1.6. *Denote*

$$\mathrm{IL} : \mathcal{I}nd(\mathcal{B}an) \to \mathcal{T}c$$

the functor defined by

$$\mathrm{IL}(\text{``}\varinjlim_{i \in \mathcal{I}}\text{''}\, E_i) = \varinjlim_{i \in \mathcal{I}} E_i.$$

Let E be an object of $\mathcal{I}nd(\mathcal{B}an)$ and let F be a complete object of $\mathcal{T}c$. Then,

$$\mathrm{Hom}_{\mathcal{I}nd(\mathcal{B}an)}(E, \mathrm{IB}(F)) \simeq \mathrm{Hom}_{\mathcal{T}c}(\mathrm{IL}(E), F).$$

Proof. Assuming $E \simeq \text{``}\varinjlim_{i \in \mathcal{I}}\text{''}\, E_i$, we have

$$\mathrm{Hom}_{\mathcal{I}nd(\mathcal{B}an)}(E, \mathrm{IB}(F)) \simeq \varprojlim_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{I}nd(\mathcal{B}an)}(\text{``}E_i\text{''}, \mathrm{IB}(F))$$

$$\simeq \varprojlim_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{I}nd(\mathcal{B}an)}(\mathrm{IB}(E_i), \mathrm{IB}(F))$$

$$\simeq \varprojlim_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{T}c}(E_i, F) \simeq \mathrm{Hom}_{\mathcal{T}c}(\mathrm{IL}(E), F)$$

where the second isomorphism follows from Remark 1.2 and the third from Proposition 1.5. $\qquad\square$

Corollary 1.7. *Let $\mathcal{I}$ be a small category. For any object X of $\mathcal{T}c^{\mathcal{I}^{\mathrm{op}}}$ such that $X(i)$ is complete for any $i \in \mathcal{I}$, we have*

$$\mathrm{IB}(\varprojlim_{i \in \mathcal{I}} X(i)) \simeq \varprojlim_{i \in \mathcal{I}} \mathrm{IB}(X(i)).$$

Proof. For any object E of $\mathcal{I}nd(\mathcal{B}an)$, we have

$$\mathrm{Hom}_{\mathcal{I}nd(\mathcal{B}an)}(E, \mathrm{IB}(\varprojlim_{i \in \mathcal{I}} X(i))) \simeq \mathrm{Hom}_{\mathcal{T}c}(\mathrm{IL}(E), \varprojlim_{i \in \mathcal{I}} X(i))$$

$$\simeq \varprojlim_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{T}c}(\mathrm{IL}(E), X(i))$$

$$\simeq \varprojlim_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{I}nd(\mathcal{B}an)}(E, \mathrm{IB}(X(i)))$$

where the first and last isomorphisms follow from Proposition 1.6. The conclusion follows from the theory of representable functors. $\qquad\square$

Proposition 1.8. *Assume that $(F_n, f_{m,n})_{n\in\mathbb{N}}$ is an inductive system of Fréchet spaces with injective transition morphisms and that*

$$\varinjlim_{n\in\mathbb{N}} F_n$$

is complete. Then, the canonical morphism

$$\varinjlim_{n\in\mathbb{N}} \mathrm{IB}(F_n) \longrightarrow \mathrm{IB}(\varinjlim_{n\in\mathbb{N}} F_n)$$

is an isomorphism.

Proof. Applying IB to the canonical morphisms

$$r_n : F_n \longrightarrow \varinjlim_{n\in\mathbb{N}} F_n$$

and using the characterization of inductive limits, we get the canonical morphism

$$\varinjlim_{n\in\mathbb{N}} \mathrm{IB}(F_n) \longrightarrow \mathrm{IB}(\varinjlim_{n\in\mathbb{N}} F_n). \tag{$*$}$$

Let B be a closed absolutely convex bounded subset of $\varinjlim_{n\in\mathbb{N}} F_n$. It follows from e.g. [6, Chap. IV, § 19, 5.(5) (p. 225)] that, for some $n \in \mathbb{N}$, B is the image of a closed absolutely convex bounded subset B_n of F_n by the canonical morphism r_n. Since r_n is injective, it induces the isomorphism of semi-normed spaces

$$(F_n)_{B_n} \xrightarrow{\sim} (\varinjlim_{n\in\mathbb{N}} F_n)_B.$$

Hence, we get the isomorphism of Banach spaces

$$(\widehat{\varinjlim_{n\in\mathbb{N}} F_n})_B \xrightarrow{\sim} \widehat{(F_n)}_{B_n}.$$

Composing with the morphism

$$\text{``}\widehat{(F_n)}_{B_n}\text{''} \longrightarrow \mathrm{IB}(F_n) \longrightarrow \varinjlim_{n\in\mathbb{N}} \mathrm{IB}(F_n),$$

we get a canonical morphism

$$\text{``}(\widehat{\varinjlim_{n\in\mathbb{N}} F_n})_B\text{''} \longrightarrow \varinjlim_{n\in\mathbb{N}} \mathrm{IB}(F_n).$$

Finally, using the characterization of inductive limits, we obtain a canonical morphism

$$\mathrm{IB}(\varinjlim_{n\in\mathbb{N}} F_n) = \varinjlim_{B\in\mathcal{B}_{\varinjlim F_n}} \text{``}(\widehat{\varinjlim_{n\in\mathbb{N}} F_n})_B\text{''} \longrightarrow \varinjlim_{n\in\mathbb{N}} \mathrm{IB}(F_n).$$

A direct computation shows that this morphism is a left and right inverse of $(*)$. $\square$

Remark 1.9. Note that, thanks to [8, Proposition 7.2] and [8, Corollary 7.2], a countable filtering inductive system of Fréchet spaces which is $\varinjlim$-acyclic in $\mathcal{T}c$ is essentially equivalent to an inductive system which satisfies the assumptions of the preceding proposition. Hence, IB also commutes with the inductive limit functor in such a situation.

Definition 1.10. Let E and F be two objects of $\mathcal{T}c$. As usual, we denote by $\mathrm{L}_b(E, F)$ the vector space $\mathrm{Hom}_{\mathcal{T}c}(E, F)$ endowed with the system of semi-norms

$$\{p_B : p \text{ continuous semi-norm of } F,\ B \text{ bounded subset of } E\}$$

where

$$p_B(f) = \sup_{e \in B} p(f(e)).$$

Lemma 1.11. *Let E and F be two objects of $\mathcal{T}c$. Assume E is bornological. Then,*

$$\mathrm{L}_b(E, F) \simeq \varprojlim_{B \in \mathcal{B}_E} \mathrm{L}_b(E_B, F)$$

in $\mathcal{T}c$. Assume moreover that F is complete. Then,

$$\mathrm{L}_b(E, F) \simeq \varprojlim_{B \in \mathcal{B}_E} \mathrm{L}_b(\widehat{E}_B, F)$$

in $\mathcal{T}c$.

Proof. Keeping in mind the properties of bornological spaces, it is clear from the definition of $\mathrm{L}_b(E, F)$ that

$$\mathrm{L}_b(E, F) \simeq \varprojlim_{B \in \mathcal{B}_E} \mathrm{L}_b(E_B, F).$$

Since any ball of $\widehat{E}_B$ is included in the closure of a semi-ball of E_B, any bounded subset of $\widehat{E}_B$ is included in the closure of a bounded subset of E_B. This property and the completeness of F shows that

$$\mathrm{L}_b(E_B, F) \simeq \mathrm{L}_b(\widehat{E}_B, F).$$

Hence the conclusion.   □

Lemma 1.12. *If E is a Banach space and if F is a complete object of $\mathcal{T}c$, then*

$$\mathrm{IB}(\mathrm{L}_b(E, F)) \simeq \mathrm{L}\left(\text{``}E\text{''}, \mathrm{IB}(F)\right).$$

Proof. For any $B' \in \mathcal{B}_F$, set

$$B'_b = \{f \in \mathrm{Hom}_{\mathcal{T}c}(E, F) : \|e\| \leq 1 \implies f(e) \in B'\}.$$

Clearly, B'_b belongs to $\mathcal{B}_{\mathrm{L}_b(E,F)}$. Moreover, if B' is closed in F, then B'_b is closed in $\mathrm{L}_b(E, F)$ and one checks easily that

$$(\mathrm{L}_b(E, F))_{B'_b} \simeq \mathrm{L}(E, F_{B'})$$

as Banach spaces. Hence, one has successively

$$\mathrm{IB}(\mathrm{L}_b(E, F)) = \underset{B \in \overline{\mathcal{B}}_{\mathrm{L}_b(E,F)}}{\text{``}\varinjlim\text{''}} (\mathrm{L}_b(E, F))_B \simeq \underset{B' \in \overline{\mathcal{B}}_F}{\text{``}\varinjlim\text{''}} (\mathrm{L}_b(E, F))_{B'_b}$$

$$\simeq \underset{B' \in \overline{\mathcal{B}}_F}{\text{``}\varinjlim\text{''}} \mathrm{L}(E, F_{B'}) \simeq \mathrm{L}(\text{``}E\text{''}, \mathrm{IB}(F))$$

where the second isomorphism follows from the fact that the inclusion

$$\{B'_b : B' \in \overline{\mathcal{B}}_F\} \subset \overline{\mathcal{B}}_{\mathrm{L}_b(E,F)}$$

is cofinal. $\square$

Proposition 1.13. *Let E and F be two objects of $\mathcal{T}c$. Assume E bornological and F complete. Then,*

$$\mathrm{IB}(\mathrm{L}_b(E, F)) \simeq \mathrm{L}(\mathrm{IB}(E), \mathrm{IB}(F)).$$

Proof. We have successively

$$\mathrm{IB}(\mathrm{L}_b(E, F)) \simeq \mathrm{IB}(\underset{B \in \mathcal{B}_E}{\varprojlim} \mathrm{L}_b(\widehat{E}_B, F)) \tag{1}$$

$$\simeq \underset{B \in \mathcal{B}_E}{\varprojlim} \mathrm{IB}(\mathrm{L}_b(\widehat{E}_B, F)) \tag{2}$$

$$\simeq \underset{B \in \mathcal{B}_E}{\varprojlim} \mathrm{L}(\text{``}\widehat{E}_B\text{''}, \mathrm{IB}(F)) \tag{3}$$

$$\simeq \mathrm{L}(\mathrm{IB}(E), \mathrm{IB}(F)),$$

where the isomorphism (1) follows from Lemma 1.11, (2) from Corollary 1.7 and (3) from Lemma 1.12. $\square$

Remark 1.14. (1) Let E, F, G be three objects of $\mathcal{T}c$. Recall that a bilinear application

$$b : E \times F \to G$$

is continuous if and only if for any continuous semi-norm r of G, there are continuous semi-norms p and q of E and F respectively such that

$$r(b(x, y)) \leq p(x)q(y).$$

(2) Let E, F be two objects of $\mathcal{T}c$ with P and Q as systems of semi-norms. As usual, if $p \in P$ and $q \in Q$, we denote $p \otimes q$ the semi-norm on $E \otimes F$ defined by

$$(p \otimes q)(u) = \inf_{u = \sum x_i \otimes y_i} \sum_i p(x_i)q(y_i).$$

Recall that $E \otimes_\pi F$ is the object of $\mathcal{T}c$ obtained by endowing $E \otimes F$ with the system of semi-norms induced by

$$\{p \otimes q : p \in P, q \in Q\}.$$

From this definition, it follows immediately that any continuous bilinear map

$$b : E \times F \rightarrow G$$

factors uniquely through a continuous linear map

$$E \otimes_\pi F \rightarrow G.$$

Proposition 1.15. *There is a canonical morphism*

$$\mathrm{IB}(E) \,\hat{\otimes}\, \mathrm{IB}(F) \rightarrow \mathrm{IB}(E \otimes_\pi F).$$

Proof. Note that if $B \in \mathcal{B}_E$ and $B' \in \mathcal{B}_F$, then

$$B \otimes B' = \langle \{b \otimes b' : b \in B, b' \in B'\} \rangle$$

is a bounded absolutely convex subset of $E \otimes F$. As a matter of fact,

$$(p \otimes q)(b \otimes b') \leq p(b)q(b') \leq \sup_{b \in B} p(b) \sup_{b' \in B'} q(b').$$

Moreover, we have a canonical linear map

$$E_B \otimes F_{B'} \rightarrow (E \otimes_\pi F)_{B \otimes B'}.$$

This map is clearly continuous since $e \otimes f \in B \otimes B'$ when $e \in B$, $e' \in B'$. Applying the completion functor, we get a morphism

$$\widehat{E}_B \,\hat{\otimes}\, \widehat{F}_{B'} \rightarrow (\widehat{E \otimes_\pi F})_{B \otimes B'}$$

and hence a morphism

$$\text{``}\widehat{E}_B\text{''} \,\hat{\otimes}\, \text{``}\widehat{F}_{B'}\text{''} \simeq \mathrm{IB}(\widehat{E}_B \,\hat{\otimes}\, \widehat{F}_{B'}) \rightarrow \mathrm{IB}(E \otimes_\pi F).$$

Using the definition of inductive limits, we get a morphism

$$\mathrm{IB}(E) \,\hat{\otimes}\, \mathrm{IB}(F) \simeq \varinjlim_{B \in \mathcal{B}_E} \varinjlim_{B' \in \mathcal{B}_F} \text{``}\widehat{E}_B\text{''} \,\hat{\otimes}\, \text{``}\widehat{F}_{B'}\text{''} \rightarrow \mathrm{IB}(E \otimes_\pi F).$$

$\square$

2   Some acyclicity results for L and $\hat{\otimes}$ in $\mathcal{I}nd(\mathcal{B}an)$

Lemma 2.1. *Let X, Y be two Banach spaces and let $f : X \to Y$ be a nuclear map. Then, there is a continuous linear map $p : X \to c^0$ and a nuclear map $c : c^0 \to Y$ making the diagram*

$$
\begin{array}{ccc}
 & c^0 & \\
{\scriptstyle p}\nearrow & & \searrow{\scriptstyle c} \\
X & \xrightarrow{\quad f \quad} & Y
\end{array}
$$

commutative.

Proof. Since $f : X \to Y$ is nuclear, there is a bounded sequence x_n^* of $\mathrm{D}(X)$, a bounded sequence y_n of Y and a summable sequence λ_n of complex numbers such that

$$
f(x) = \sum_{n=0}^{+\infty} \lambda_n \langle x_n^*, x \rangle \, y_n \qquad \forall x \in X.
$$

Since λ_n is summable, one can find a sequence r_n of non-zero complex numbers converging to zero such that λ_n/r_n is still summable. One checks easily that the maps $p : X \to c^0$ and $c : c^0 \to Y$ defined by

$$
p(x)_n = r_n \langle x_n^*, x \rangle \qquad \text{and} \qquad c(s) = \sum_{n=0}^{+\infty} \frac{\lambda_n}{r_n} y_n s_n
$$

have the requested properties. $\qquad\square$

Definition 2.2. *A projective system $E : I^{\mathrm{op}} \to \mathcal{B}an$ where I is a filtering ordered set is* nuclear *if for any $i \in I$, there is $j \in I$, $j \geq i$ such that the transition morphism*

$$
e_{i,j} : E_j \to E_i
$$

is nuclear.

Lemma 2.3. *Let I be an infinite filtering ordered set and let $E : I^{\mathrm{op}} \to \mathcal{B}an$ be a nuclear projective system. Then,*

$$
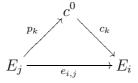\text{``}\varprojlim_{i \in I}\text{''} \, E_i \simeq \text{``}\varprojlim_{k \in K}\text{''} \, X_k.
$$

where $X : K^{\mathrm{op}} \to \mathcal{B}an$ is a projective system with nuclear transition morphisms such that $X^k = c^0$ for any $k \in K$ and $\#K = \#I$.

Proof. Consider the set

$$K = \{(i,j) \in I \times I : j \geq i,\ e_{i,j} : E_j \longrightarrow E_i \text{ nuclear}\}.$$

The relation "$\geq$" defined by setting $(i', j') \geq (i, j)$ if $(i', j') = (i, j)$ or $i' \geq j$ turns K into a filtering ordered set. By Lemma 2.1, for any $k = (i, j) \in K$, we may choose a continuous linear map $p_k : E_j \longrightarrow c^0$ and a nuclear map $c_k : c^0 \longrightarrow E_i$ making the diagram

$$
\begin{array}{ccc}
 & c^0 & \\
{\scriptstyle p_k}\nearrow & & \searrow{\scriptstyle c_k} \\
E_j & \xrightarrow[\ e_{i,j}\ ]{} & E_i
\end{array}
$$

commutative. For any $k \in K$, we set $X_k = c^0$ and $x_{k,k} = \mathrm{id}_{X_k}$. If $k' = (i', j') > k = (i, j)$, we set

$$x_{k,k'} = p_k \circ e_{j,i'} \circ c_{k'} : X_{k'} \longrightarrow X_k.$$

The map $c_{k'}$ being nuclear, $x_{k,k'}$ is also nuclear. An easy computation shows that if $k < k' < k''$, then $x_{k,k'} \circ x_{k',k''} = x_{k,k''}$. Consider the functors

$$\Phi : K \longrightarrow I \qquad \text{and} \qquad \Psi : K \longrightarrow I$$

defined by $\Phi((i,j)) = i$ and $\Psi((i,j)) = j$. They are clearly cofinal and if $k' \geq k$ in K, the diagrams

$$
\begin{array}{ccc}
X_{k'} & \xrightarrow{\ c_{k'}\ } & E_{\Phi(k')} \\
{\scriptstyle x_{k,k'}}\downarrow & & \downarrow{\scriptstyle e_{\Phi(k),\Phi(k')}} \\
X_k & \xrightarrow[\ c_k\ ]{} & E_{\Phi(k)}
\end{array}
\qquad\qquad
\begin{array}{ccc}
E_{\Psi(k')} & \xrightarrow{\ p_{k'}\ } & X_{k'} \\
{\scriptstyle e_{\Psi(k),\Psi(k')}}\downarrow & & \downarrow{\scriptstyle x_{k,k'}} \\
E_{\Psi(k)} & \xrightarrow[\ p_k\ ]{} & X_k
\end{array}
$$

are commutative. Hence, we get the two morphisms

$$\text{``}\varprojlim_{k \in K}\text{''}\, X_k \longrightarrow \text{``}\varprojlim_{k \in K}\text{''}\, E_{\Phi(k)} \simeq \text{``}\varprojlim_{i \in I}\text{''}\, E_i \quad \text{and} \quad \text{``}\varprojlim_{j \in I}\text{''}\, E_j \simeq \text{``}\varprojlim_{k \in K}\text{''}\, E_{\Psi(k)} \longrightarrow \text{``}\varprojlim_{k \in K}\text{''}\, X_k.$$

Since these morphisms are easily checked to be inverse one of each other, the proof is complete. $\qquad\square$

Remark 2.4. Hereafter, as usual, we denote e_n the element of c^0 defined by

$$(e_n)_m = \delta_{n,m}$$

and we denote e_n^* the element of $\mathrm{D}(c^0)$ defined by

$$\langle e_n^*, x \rangle = x_n.$$

Lemma 2.5. *For any Banach space Y and any nuclear map*

$$u : c^0 \longrightarrow Y$$

the sequence $\|u(e_n)\|_Y$ is summable and for any $x \in c^0$, we have

$$u(x) = \sum_{n=0}^{+\infty} \langle e_n^*, x \rangle \, u(e_n).$$

Proof. Since u is nuclear, we can find a bounded sequence x_n^* of $\mathrm{D}(c^0)$, a bounded sequence y_n of Y and a summable sequence λ_n of complex numbers such that

$$u(x) = \sum_{n=0}^{+\infty} \lambda_n \, \langle x_n^*, x \rangle \, y_n$$

for any $x \in c^0$. Using the isomorphism $\mathrm{D}(c^0) \simeq l^1$, we see that

$$\sum_{m=0}^{+\infty} | \, \langle x_n^*, e_m \rangle \, | = \|x_n^*\|_{\mathrm{D}(c^0)} . \tag{*}$$

Therefore,

$$\begin{aligned}
\sum_{m=0}^{M} \|u(e_m)\| &\leq \sum_{m=0}^{M} \sum_{n=0}^{+\infty} |\lambda_n| | \, \langle x_n^*, e_m \rangle \, | \, \|y_n\|_Y \\
&\leq \sum_{n=0}^{+\infty} |\lambda_n| \, \|x_n^*\|_{\mathrm{D}(c^0)} \, \|y_n\|_Y \\
&\leq \left(\sum_{n=0}^{+\infty} |\lambda_n| \right) \sup_{n \in \mathbb{N}} \|x_n^*\|_{\mathrm{D}(c^0)} \sup_{n \in \mathbb{N}} \|y_n\|_Y
\end{aligned}$$

and the sequence $\|u(e_n)\|_Y$ is summable. Moreover,

$$\begin{aligned}
u(x) &= \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \lambda_n \, \langle x_n^*, e_m \rangle \, x_m y_n \\
&= \sum_{m=0}^{+\infty} x_m \left(\sum_{n=0}^{+\infty} \lambda_n \, \langle x_n^*, e_m \rangle \, y_n \right) \\
&= \sum_{m=0}^{+\infty} \langle e_m^*, x \rangle \, u(e_m)
\end{aligned}$$

where the permutation of the sums is justified using (*). $\qquad\square$

Lemma 2.6. *Let I be an infinite filtering ordered set and let $X : I^{\mathrm{op}} \to \mathcal{B}an$ be a nuclear projective system. Assume Y is a Fréchet space. Then, the morphisms*

$$\varphi_i : X_i \,\hat{\otimes}_\pi\, Y \longrightarrow \mathrm{L}_b(\mathrm{D}(X_i), Y)$$

defined by setting

$$\varphi_i(x \,\hat{\otimes}_\pi\, y)(x^*) = \langle x^*, x \rangle\, y \qquad \forall x \in X_i,\ y \in Y,\ x^* \in \mathrm{D}(X_i)$$

induce an isomorphism

$$\text{``}\varprojlim_{i \in I}\text{''}\, X_i \,\hat{\otimes}_\pi\, Y \simeq \text{``}\varprojlim_{i \in I}\text{''}\, \mathrm{L}_b(\mathrm{D}(X_i), Y).$$

In particular, for $Y = \mathbb{C}$, we have

$$\text{``}\varprojlim_{i \in I}\text{''}\, X_i \simeq \text{``}\varprojlim_{i \in I}\text{''}\, \mathrm{D}(\mathrm{D}(X_i)).$$

Proof. By Lemma 2.3, we may assume that $X_i = c^0$ for any $i \in I$ and that the transition morphisms

$$x_{i,j} : X_j \longrightarrow X_i \qquad (j > i)$$

are nuclear.

One checks easily that φ_i is a well-defined continuous map. By Lemma 2.5, we know that the sequence $(\|x_{i,j}(e_n)\|_{X_i})_{n \in \mathbb{N}}$ is summable and that

$$x_{i,j}(c) = \sum_{n=0}^{+\infty} \langle e_n^*, c \rangle\, x_{i,j}(e_n) \qquad \forall c \in X_j.$$

Therefore, we may define a continuous linear map

$$\psi_{i,j} : \mathrm{L}_b(\mathrm{D}(X_j), Y) \longrightarrow X_i \,\hat{\otimes}_\pi\, Y$$

by setting

$$\psi_{i,j}(h) = \sum_{n=0}^{+\infty} x_{i,j}(e_n) \,\hat{\otimes}_\pi\, h(e_n^*).$$

One sees easily that the morphisms φ_i and $\psi_{i,j}$ induce morphisms of pro-objects

$$\text{``}\varprojlim_{i \in I}\text{''}\, X_i \,\hat{\otimes}_\pi\, Y \longrightarrow \text{``}\varprojlim_{i \in I}\text{''}\, \mathrm{L}_b(\mathrm{D}(X_i), Y)$$

and

$$\text{``}\varprojlim_{i \in I}\text{''}\, \mathrm{L}_b(\mathrm{D}(X_i), Y) \longrightarrow \text{``}\varprojlim_{i \in I}\text{''}\, X_i \,\hat{\otimes}_\pi\, Y.$$

A direct computation shows that these morphisms are inverse one of each other. $\square$

Definition 2.7. We say that a filtering projective system $E : I^{\mathrm{op}} \to \mathcal{T}c$ satisfies *Condition ML* if for any $i \in I$, any semi-norm p of E_i and any $\epsilon > 0$, there is $i' \geq i$ such that

$$e_{i,i'}(E_i) \subset b_p(\epsilon) + e_{i,i''}(E_{i''}) \qquad \forall i'' \geq i'.$$

Remark 2.8. By [13, Proposition 1.2.9] (which is a direct consequence of [12, Theorem 4.6]), a countable filtering projective system of Fréchet spaces is $\varprojlim$-acyclic in $\mathcal{T}c$ if and only if it satisfies Condition ML.

Lemma 2.9. *Let $E : I^{\mathrm{op}} \to \mathcal{T}c$ and $F : J^{\mathrm{op}} \to \mathcal{T}c$ be two filtering projective systems. If E and F satisfy Condition ML, then the projective system*

$$E \mathbin{\hat{\otimes}_\pi} F : (I \times J)^{\mathrm{op}} \to \mathcal{F}r$$

defined by

$$(E \mathbin{\hat{\otimes}_\pi} F)(i,j) = E_i \mathbin{\hat{\otimes}_\pi} F_j$$

satisfies Condition ML.

Proof. Let $(i,j) \in I \times J$ and let $p \mathbin{\hat{\otimes}_\pi} q$ be a semi-norm of $E_i \mathbin{\hat{\otimes}_\pi} F_j$. It follows from our assumptions, that there is $i' \geq i$ and $j' \geq j$ such that

$$e_{i,i'}(E_{i'}) \subset b_p(1) + e_{i,i''}(E_{i''}) \qquad \forall i'' \geq i' \tag{*}$$

and

$$f_{j,j'}(F_{j'}) \subset b_q(1) + f_{j,j''}(F_{j''}) \qquad \forall j'' \geq j'. \tag{**}$$

Fix $(i'',j'') \geq (i',j')$. Since the maps $e_{i,i'}$, $f_{j,j'}$ and $e_{i,i''}$ are continuous, we can find a semi-norm p' of $E_{i'}$, a semi-norm q' of $F_{j'}$ and a semi-norm p'' of $E_{i''}$ such that

$$p \circ e_{i,i'} \leq p', \qquad q \circ f_{j,j'} \leq q' \qquad \text{and} \qquad p \circ e_{i,i''} \leq p''.$$

Consider $\epsilon > 0$ and let z' be an element of $E_{i'} \otimes_\pi F_{j'}$ of the type $x' \otimes_\pi y'$ where $x' \in E_{i'}$, $y' \in F_{j'}$. Using (*) and (**) above, we obtain $x'' \in E_{i''}$ and $y'' \in F_{j''}$ such that

$$p(e_{i,i'}(x') - e_{i,i''}(x'')) \leq \frac{\epsilon}{2(1 + q'(y'))} \quad \text{and} \quad q(f_{j,j'}(y') - f_{j,j''}(y'')) \leq \frac{\epsilon}{2(1 + p''(x''))}.$$

For $z'' = x'' \otimes_\pi y'' \in E_{i''} \otimes_\pi F_{j''}$, we get

$$\begin{aligned}
(p \otimes_\pi q) &\left((e_{i,i'} \otimes_\pi f_{j,j'})(z') - (e_{i,i''} \otimes_\pi f_{j,j''})(z'') \right) \\
&= (p \otimes_\pi q) \left((e_{i,i'}(x') - e_{i,i''}(x'')) \otimes_\pi f_{j,j'}(y') + e_{i,i''}(x'') \otimes_\pi (f_{j,j'}(y') - f_{j,j''}(y'')) \right) \\
&\leq p(e_{i,i'}(x') - e_{i,i''}(x''))q(f_{j,j'}(y')) + p(e_{i,i''}(x''))q(f_{j,j'}(y') - f_{j,j''}(y'')) \leq \epsilon.
\end{aligned}$$

Since any element of $E_i \otimes_\pi F_j$ is a finite sum of elements of the type considered above, we see that for any $\epsilon > 0$,

$$(e_{i,i'} \otimes_\pi f_{j,j'})(E_{i'} \otimes_\pi F_{j'}) \subset b_{p \otimes_\pi q}(\epsilon) + (e_{i,i''} \otimes_\pi f_{j,j''})(E_{i''} \otimes_\pi F_{j''}).$$

The conclusion follows directly since $E_{i'} \otimes_\pi F_{j'}$ is dense in $E_{i'} \hat{\otimes}_\pi F_{j'}$. $\qquad\square$

Remark 2.10. Let E be an object of $\mathcal{T}c$. Recall that E is of type FN if it is a nuclear Fréchet space and that E is of type DFN if it is isomorphic to the strong dual of a nuclear Fréchet space.

Lemma 2.11. *Assume X is a FN space. Then, there is a projective system*

$$(X_n, x_{n,m})_{n \in \mathbb{N}}$$

of Banach spaces such that

(a) *there is an isomorphism*

$$X \simeq \varprojlim_{n \in \mathbb{N}} X_n;$$

(b) *for $m > n$, the transition map*

$$x_{n,m} : X_m \longrightarrow X_n$$

 is nuclear and has a dense range;

(c) *there is an isomorphism*

$$\mathrm{D_b}(X) \simeq \varinjlim_{n \in \mathbb{N}} \mathrm{D}(X_n);$$

(d) *for $m > n$, the transition map*

$$\mathrm{D}(x_{n,m}) : \mathrm{D}(X_n) \longrightarrow \mathrm{D}(X_m)$$

 is nuclear and injective.

Proof. Since X is a FN space, there is a cofinal increasing sequence $(p_n)_{n \in \mathbb{N}}$ of continuous semi-norms of X such that the canonical map

$$X_{p_{n+1}} \longrightarrow X_{p_n}$$

is nuclear. For such a sequence, the canonical map

$$\widehat{X}_{p_{n+1}} \longrightarrow \widehat{X}_{p_n}$$

is also nuclear and has a dense range. Moreover, it is well-known (see e.g. [6, Chap. IV, § 19, 9.(1) (p. 231)]) that

$$X \simeq \varprojlim_{n \in \mathbb{N}} \widehat{X}_{p_n}.$$

Clearly,

$$\mathrm{D_i}(X) \simeq \varinjlim_{n \in \mathbb{N}} \mathrm{D}(X_{p_n}) \simeq \varinjlim_{n \in \mathbb{N}} \mathrm{D}(\widehat{X}_{p_n})$$

where $\mathrm{D_i}(X)$ is the inductive dual of X.

Recall that an absolutely convex subset V is a neighborhood of 0 in $\mathrm{D_i}(X)$ if it absorbs any equicontinuous subset of X'. Hence, it is clear that a neighborhood of 0 in $\mathrm{D_b}(X)$ is a neighborhood of 0 in $\mathrm{D_i}(X)$. We know that X is reflexive (see e.g. [9, § 5.3.2 (p. 93)]). Hence, $\mathrm{D_b}(X)$ is bornological (see e.g. [6, Chap. VI, § 29, 4.(4) (p. 400)]). The space X being itself bornological, the bounded subsets of $\mathrm{D_b}(X)$ are equicontinuous. So, any neighborhood of 0 in $\mathrm{D_i}(X)$ is a neighborhood of 0 in $\mathrm{D_b}(X)$ and $\mathrm{D_i}(X) \simeq \mathrm{D_b}(X)$.

Since (d) follows directly from (b), the proof is complete. $\qquad\square$

Proposition 2.12. *Assume E is a DFN space and F is a Fréchet space. Then, the canonical morphism*

$$\mathrm{Hom}\,(\mathrm{IB}(E), \mathrm{IB}(F)) \longrightarrow \mathrm{RHom}\,(\mathrm{IB}(E), \mathrm{IB}(F))$$

is an isomorphism.

Proof. Since E is a DFN space, there is a FN space X such that

$$E \simeq \mathrm{D_b}(X).$$

Let $(X_n, x_{n,m})$ be a projective system of the kind considered in Lemma 2.11. We have

$$E \simeq \mathrm{D_b}(X) \simeq \varinjlim_{n \in \mathbb{N}} \mathrm{D}(X_n).$$

Since the transition morphisms

$$\mathrm{D}(x_{n,m}) : \mathrm{D}(X_n) \longrightarrow \mathrm{D}(X_m) \qquad (m > n)$$

are injective and E is complete, Proposition 1.8 and Remark 1.2 show that

$$\mathrm{IB}(E) \simeq \varinjlim_{n \in \mathbb{N}} \text{``}\mathrm{D}(X_n)\text{''}.$$

Using Lemma 2.3, we find a nuclear projective system $(Y_n, y_{n,m})$ with $Y_n = c^0$ such that

$$\text{``}\varprojlim_{n \in \mathbb{N}}\text{''}\, X_n \simeq \text{``}\varprojlim_{n \in \mathbb{N}}\text{''}\, Y_n.$$

It follows that

$$\text{IB}(E) \simeq \varinjlim_{n \in \mathbb{N}} \text{``D}(Y_n)\text{''}.$$

Hence, we have successively

$$\text{RHom}\,(\text{IB}(E), \text{IB}(F)) \simeq \text{RHom}\,(\text{L}\varinjlim_{n \in \mathbb{N}} \text{``D}(Y_n)\text{''}, \text{IB}(F)) \tag{1}$$

$$\simeq \text{R}\varprojlim_{n \in \mathbb{N}} \text{RHom}\,(\text{``D}(Y_n)\text{''}, \text{IB}(F)) \tag{2}$$

$$\simeq \text{R}\varprojlim_{n \in \mathbb{N}} \text{Hom}\,(\text{``D}(Y_n)\text{''}, \text{IB}(F)) \tag{3}$$

$$\simeq \text{R}\varprojlim_{n \in \mathbb{N}} \text{Hom}\,(\text{IB}(\text{D}(Y_n)), \text{IB}(F)) \tag{4}$$

$$\simeq \text{R}\varprojlim_{n \in \mathbb{N}} \text{Hom}_{\mathcal{T}c}(\text{D}(Y_n), F) \tag{5}$$

where the isomorphism (1) follows from the fact that filtering inductive limits are exact in $\mathcal{I}nd(\mathcal{B}an)$, (2) follows from [11, Proposition 3.6.3], (3) follows from the fact that $\text{``D}(Y_n)\text{''} \simeq \text{``D}(c^0)\text{''} \simeq \text{``}l^1\text{''}$ is projective in $\mathcal{I}nd(\mathcal{B}an)$, (4) follows from Remark 1.2 and (5) follows from Proposition 1.5. By Lemma 2.6, we have the isomorphism

$$\text{``}\varprojlim_{n \in \mathbb{N}}\text{''}\,(Y_n \,\hat{\otimes}_\pi F) \simeq \text{``}\varprojlim_{n \in \mathbb{N}}\text{''}\, \text{L}_b(\text{D}(Y_n), F).$$

Forgetting the topologies and applying the derived projective limit functor for pro-objects (see [11]), we obtain the isomorphism

$$\text{R}\varprojlim_{n \in \mathbb{N}}(Y_n \,\hat{\otimes}_\pi F) \simeq \text{R}\varprojlim_{n \in \mathbb{N}} \text{Hom}_{\mathcal{T}c}(\text{D}(Y_n), F).$$

Since $(X_n, x_{n,m})_{n \in \mathbb{N}}$ satisfies Condition ML, it is $\varprojlim$-acyclic in $\mathcal{T}c$ (see Remark 2.8). It follows that $(Y_n, y_{n,m})_{n \in \mathbb{N}}$ is also $\varprojlim$-acyclic in $\mathcal{T}c$ and, hence, satisfies Condition ML. Using Lemma 2.9, we see that $\text{R}\varprojlim_{n \in \mathbb{N}}(Y_n \,\hat{\otimes}_\pi F)$ is concentrated in degree 0. It follows that the projective system

$$\left(\text{Hom}_{\mathcal{T}c}(\text{D}(Y_n), F)\right)_{n \in \mathbb{N}}$$

is $\varprojlim$-acyclic and the conclusion follows. $\square$

Theorem 2.13. *Assume E is a DFN space and F is a Fréchet space. Then, the canonical morphism*

$$\mathrm{L}\left(\mathrm{IB}(E), \mathrm{IB}(F)\right) \to \mathrm{RL}\left(\mathrm{IB}(E), \mathrm{IB}(F)\right)$$

is an isomorphism.

Proof. It is sufficient to show that

$$LH^k(\mathrm{RL}\left(\mathrm{IB}(E), \mathrm{IB}(F)\right)) \simeq 0$$

for $k > 0$. This will be the case if

$$\mathrm{Hom}\left(\text{``}l^1(I)\text{''}, \mathrm{RL}\left(\mathrm{IB}(E), \mathrm{IB}(F)\right)\right)$$

is concentrated in degree 0 for any set I.

Let I be an arbitrary set. Since "$l^1(I)$" is a projective object of $\mathcal{I}nd(\mathcal{B}an)$ and since F is complete, we have

$$\begin{aligned}
\mathrm{RL}\left(\text{``}l^1(I)\text{''}, \mathrm{IB}(F)\right) &\simeq \mathrm{L}\left(\mathrm{IB}(l^1(I)), \mathrm{IB}(F)\right) \\
&\simeq \mathrm{IB}(\mathrm{L}_b(l^1(I), F)) \\
&\simeq \mathrm{IB}(l^\infty(I, F))
\end{aligned}$$

where $l^\infty(I, F)$ is the Fréchet space formed by the bounded families $(x_i)_{i \in I}$ of F (a fundamental system of semi-norms being given by

$$\{p_I : p \text{ continuous semi-norm of } F\}$$

where $p_I((x_i)_{i \in I}) = \sup_{i \in I} p(x_i))$. Therefore, we have the chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}\left(\text{``}l^1(I)\text{''}, \mathrm{RL}\left(\mathrm{IB}(E), \mathrm{IB}(F)\right)\right) &\simeq \mathrm{RHom}\left(\text{``}l^1(I)\text{''}, \mathrm{RL}\left(\mathrm{IB}(E), \mathrm{IB}(F)\right)\right) \\
&\simeq \mathrm{RHom}\left(\text{``}l^1(I)\text{''} \,\hat{\otimes}^L\, \mathrm{IB}(E), \mathrm{IB}(F)\right) \\
&\simeq \mathrm{RHom}\left(\mathrm{IB}(E) \,\hat{\otimes}^L\, \text{``}l^1(I)\text{''}, \mathrm{IB}(F)\right) \\
&\simeq \mathrm{RHom}\left(\mathrm{IB}(E), \mathrm{RL}\left(\text{``}l^1(I)\text{''}, \mathrm{IB}(F)\right)\right) \\
&\simeq \mathrm{RHom}\left(\mathrm{IB}(E), \mathrm{IB}(l^\infty(I, F))\right)
\end{aligned}$$

and the conclusion follows from Proposition 2.12. $\qquad\square$

Lemma 2.14. *Let X, Y be two Banach spaces and let $f : X \to Y$ be a nuclear map. Then, there is a nuclear map $p : X \to l^1$ and a continuous linear map $c : l^1 \to Y$ making the diagram*

commutative.

Proof. Work as for Lemma 2.1. □

Definition 2.15. An inductive system $E : I \to \mathcal{B}an$ where I is a filtering ordered set is *nuclear* if for any $i \in I$, there is $j \in I$, $j \geq i$ such that the transition morphism

$$e_{j,i} : E_i \to E_j$$

is nuclear. An object of $\mathcal{I}nd(\mathcal{B}an)$ is *nuclear* if it corresponds to a nuclear inductive system.

Remark 2.16. Working as in the proof of Proposition 2.12, we see easily that $\text{IB}(E)$ is nuclear if E is a DFN space.

Lemma 2.17. *Let I be an infinite filtering ordered set and let $E : I \to \mathcal{B}an$ be a nuclear inductive system. Then,*

$$\text{``}\varinjlim_{i \in I}\text{''}\, E_i \simeq \text{``}\varinjlim_{k \in K}\text{''}\, X_k.$$

where $X : K \to \mathcal{B}an$ is an inductive system with nuclear transition morphisms such that $X^k = l^1$ for any $k \in K$ and $\#K = \#I$.

Proof. Work as for Lemma 2.3 using Lemma 2.14. □

Lemma 2.18. *Let I be a filtering ordered set. For any $F \in D^-(\mathcal{I}nd(\mathcal{B}an))$ and any $E \in D^-(\mathcal{I}nd(\mathcal{B}an)^I)$, we have*

$$(\varinjlim_{i \in I} E_i) \,\hat{\otimes}^L\, F \simeq \varinjlim_{i \in I}(E_i \,\hat{\otimes}^L\, F).$$

Proof. If $P_.$ is a projective resolution of F, we have successively

$$(\varinjlim_{i \in I} E_i) \,\hat{\otimes}^L\, F \simeq (\varinjlim_{i \in I} E_i) \,\hat{\otimes}\, P_. \simeq \varinjlim_{i \in I}(E_i \,\hat{\otimes}\, P_.) \simeq \varinjlim_{i \in I}(E_i \,\hat{\otimes}^L\, F).$$

□

Proposition 2.19. *Let E and F be objects of $\mathcal{I}nd(\mathcal{B}an)$. Assume E is nuclear. Then,*

$$E \,\hat{\otimes}^L\, F \simeq E \,\hat{\otimes}\, F.$$

Proof. Using Lemma 2.17, we may assume that

$$E = \text{``}\varinjlim_{i \in I}\text{''}\, X_i$$

where $X : I \longrightarrow \mathcal{B}an$ is a filtering inductive system with $X_i = l^1$, the transition morphisms

$$x_{j,i} : X_i \longrightarrow X_j$$

being nuclear. We may also assume that

$$F = \text{``}\varinjlim_{j \in J}\text{''}\, Y_j$$

where $Y : J \longrightarrow \mathcal{B}an$ is a filtering inductive system. Then, we have

$$E \,\hat{\otimes}^L\, F \simeq (\text{``}\varinjlim_{i \in I}\text{''}\, X_i) \,\hat{\otimes}^L\, (\text{``}\varinjlim_{j \in J}\text{''}\, Y_j)$$

$$\simeq \varinjlim_{i \in I} \varinjlim_{j \in J} \text{``}X_i\text{''} \,\hat{\otimes}^L\, \text{``}Y_j\text{''} \tag{1}$$

$$\simeq \varinjlim_{i \in I} \varinjlim_{j \in J} \text{``}X_i\text{''} \,\hat{\otimes}\, \text{``}Y_j\text{''} \tag{2}$$

$$\simeq (\text{``}\varinjlim_{i \in I}\text{''}\, X_i) \,\hat{\otimes}\, (\text{``}\varinjlim_{j \in J}\text{''}\, Y_j)$$

$$\simeq E \,\hat{\otimes}\, F$$

where the isomorphism (1) follows from Lemma 2.18 and (2) from the fact that "X_i" $\simeq$ "l^1" is projective in $\mathcal{I}nd(\mathcal{B}an)$. $\qquad\square$

3 A topological version of Cartan's Theorem B

Proposition 3.1. *Let X be a topological space with a countable basis. If F is a presheaf of Fréchet spaces on X which is a sheaf of vector spaces, then*

$$U \mapsto \mathrm{IB}(F(U)) \qquad (U \text{ open of } X)$$

is a sheaf with values in $\mathcal{I}nd(\mathcal{B}an)$.

Proof. Let U be an open subset of X and let $\mathcal{U}$ be an open covering of U. Consider the sequence

$$0 \longrightarrow F(U) \xrightarrow{\alpha} \prod_{V \in \mathcal{U}} F(V) \xrightarrow{\beta} \prod_{V,W \in \mathcal{U}} F(V \cap W) \tag{*}$$

where α and β are the continuous applications defined by

$$p_V \circ \alpha = r_{V,U} \qquad \text{and} \qquad p_{V,W} \circ \beta = r_{V \cap W, V} \circ p_V - r_{V \cap W, W} \circ p_W$$

where p_V and $p_{V,W}$ are the canonical projections and $r_{V,U}$ is the restriction map. Since F is a sheaf of vector spaces, this sequence is algebraically exact. Let us show that it is strictly exact.

(1) If $\mathcal{U}$ is countable, $F(U)$, $\prod_{V \in \mathcal{U}} F(V)$ and $\prod_{V,W \in \mathcal{U}} F(V \cap W)$ are Fréchet spaces. Then, by the homomorphism theorem, the sequence (*) is strictly exact.

(2) Assume that $\mathcal{U}$ is not countable. Since X has a countable basis, there is a countable set A of open subsets of X such that for any open V of X,

$$V = \bigcup_{k \in \mathbb{N}} U_k, \qquad U_k \in A.$$

Then, consider the countable set

$$\mathcal{V} = \{V' \in A : \ \exists\, V \in \mathcal{U} \text{ such that } V' \subset V\}.$$

For any $U' \in \mathcal{U}$, we may assume that $U' = \bigcup_{k \in \mathbb{N}} U'_k$, with $U'_k \in \mathcal{V}$. It follows that $\mathcal{V}$ covers any U' in $\mathcal{U}$ and therefore is a covering of U. Hence, by (1), the sequence

$$0 \to F(U) \xrightarrow{\alpha'} \prod_{V' \in \mathcal{V}} F(V') \xrightarrow{\beta'} \prod_{V',W' \in \mathcal{V}} F(V' \cap W')$$

is strictly exact. Now, consider a map $f : \mathcal{V} \to \mathcal{U}$ such that $V' \subset f(V')$ for any $V' \in \mathcal{V}$. Then, consider the commutative diagram

$$
\begin{array}{ccccc}
0 \longrightarrow F(U) & \xrightarrow{\ \alpha\ } & \prod_{V \in \mathcal{U}} F(V) & \xrightarrow{\ \beta\ } & \prod_{V,W \in \mathcal{U}} F(V \cap W) \\
\Big\downarrow{\rm id} & & \Big\downarrow{\gamma} & & \Big\downarrow{\delta} \\
0 \longrightarrow F(U) & \xrightarrow[\ \alpha'\ ]{} & \prod_{V' \in \mathcal{V}} F(V') & \xrightarrow[\ \beta'\ ]{} & \prod_{V',W' \in \mathcal{V}} F(V' \cap W')
\end{array}
$$

where γ and δ are respectively defined by

$$p_{V'} \circ \gamma = r_{V',f(V')} \circ p_{f(V')}$$

and

$$p_{V',W'} \circ \delta = r_{V' \cap W', f(V') \cap f(W')} \circ p_{f(V'),f(W')}.$$

To prove that the sequence (*) is strictly exact, it is sufficient to establish that α is a kernel of β. Let $h : X \to \prod_{V \in \mathcal{U}} F(V)$ be a morphism of $\mathcal{T}c$ such that $\beta \circ h = 0$. Since $\beta' \circ \gamma \circ h = \delta \circ \beta \circ h = 0$ and since α' is a kernel of β', there is a unique morphism $h' : X \to F(U)$ such that $\alpha' \circ h' = \gamma \circ h$. Set

$$h'' = h - \alpha \circ h'.$$

We clearly have $\gamma \circ h'' = 0$ and $\beta \circ h'' = 0$. Fix $V \in \mathcal{U}$. For any $V' \in \mathcal{V}$ such that $V' \subset V$, we have

$$0 = p_{V,f(V')} \circ \beta \circ h'' = r_{V \cap f(V'),V} \circ p_V \circ h'' - r_{V \cap f(V'),f(V')} \circ p_{f(V')} \circ h''.$$

It follows that

$$
\begin{aligned}
r_{V',V} \circ p_V \circ h'' &= r_{V',V \cap f(V')} \circ r_{V \cap f(V'),V} \circ p_V \circ h'' \\
&= r_{V',V \cap f(V')} \circ r_{V \cap f(V'),f(V')} \circ p_{f(V')} \circ h'' \\
&= r_{V',f(V')} \circ p_{f(V')} \circ h'' \\
&= p_{V'} \circ \gamma \circ h'' \\
&= 0.
\end{aligned}
$$

Since $\{V' \in \mathcal{V} : V' \subset V\}$ is a covering of V and since F is a sheaf of vector spaces, we get

$$p_V \circ h'' = 0 \qquad \forall V \in \mathcal{U}.$$

It follows that $h'' = 0$ and that $h = \alpha \circ h'$. Since α is injective, h' is the unique morphism of $\mathcal{T}c$ such that $h = \alpha \circ h'$. Therefore, α is a kernel of β and the sequence (*) is strictly exact.

Finally, since the functor IB preserves projective limits of complete objects of $\mathcal{T}c$ (see Corollary 1.7), the sequence

$$0 \to \mathrm{IB}(F(U)) \xrightarrow{\mathrm{IB}(\alpha)} \prod_{V \in \mathcal{U}} \mathrm{IB}(F(V)) \xrightarrow{\mathrm{IB}(\beta)} \prod_{V,W \in \mathcal{U}} \mathrm{IB}(F(V \cap W))$$

is strictly exact in $\mathcal{I}nd(\mathcal{B}an)$. Hence, the conclusion. $\qquad\square$

Definition 3.2. For short, we denote $\mathrm{IB}(F)$ the sheaf with values in $\mathcal{I}nd(\mathcal{B}an)$ associated to a presheaf F of the kind considered in Proposition 3.1.

Hereafter, X will denote a complex analytic manifold of complex dimension d_X. We denote $\mathcal{O}_X$ the sheaf of holomorphic functions on X. Recall that for any open subset U of X, $\mathcal{O}_X(U)$ has a canonical structure of FN space. Recall moreover that if V is a relatively compact open subset of U the restriction morphism

$$\mathcal{O}_X(U) \to \mathcal{O}_X(V)$$

is nuclear. In particular, if K is a compact subset of X, then

$$\mathcal{O}_X(K) \simeq \varinjlim_{\substack{U \supset K \\ U \mathrm{open}}} \mathcal{O}_X(U)$$

topologized as an inductive limit is a DFN space.

Proposition 3.3. *For any compact subset K of X, we have*

$$\Gamma(K, \mathrm{IB}(\mathcal{O}_X)) \simeq \mathrm{IB}(\mathcal{O}_X(K)).$$

Proof. We know that K has a fundamental system $(U_n)_{n \in \mathbb{N}}$ of relatively compact open neighborhoods such that

$$\overline{U}_{n+1} \subset U_n$$

for any $n \in \mathbb{N}$. Replacing, if necessary, U_n by the union of those of its connected components which meet K, we may even assume that any connected component of U_n meets K. In this case, it follows from the principle of unique continuation that the restriction

$$\mathcal{O}_X(U_n) \longrightarrow \mathcal{O}_X(U_{n+1})$$

is injective. Moreover, by cofinality,

$$\mathcal{O}_X(K) \simeq \varinjlim_{n \in \mathbb{N}} \mathcal{O}_X(U_n).$$

Hence, by Proposition 1.8, it follows that

$$\mathrm{IB}(\varinjlim_{n \in \mathbb{N}} \mathcal{O}_X(U_n)) \simeq \varinjlim_{n \in \mathbb{N}} \mathrm{IB}(\mathcal{O}_X(U_n)).$$

Since K is a taut subspace of X, a cofinality argument shows that

$$\Gamma(K, \mathrm{IB}(\mathcal{O}_X)) \simeq \varinjlim_{n \in \mathbb{N}} \Gamma(U_n, \mathrm{IB}(\mathcal{O}_X))$$

and the conclusion follows. $\qquad\square$

Hereafter, we denote $\mathcal{C}_{\infty,X}^{(p,q)}$ the sheaf of differential forms of class C_∞ and of bitype (p,q). Recall that for any open subset U of X, $\mathcal{C}_{\infty,X}^{(p,q)}(U)$ has a canonical structure of FN space. Since the conditions of Proposition 3.1 are satisfied, $\mathrm{IB}(\mathcal{C}_{\infty,X}^{(p,q)})$ is a sheaf with values in $\mathcal{I}nd(\mathcal{B}an)$.

Proposition 3.4. *The sheaf* $\mathrm{IB}(\mathcal{C}_{\infty,X}^{(p,q)})$ *is* $\Gamma(U, \cdot)$*-acyclic for any open subset U of X.*

Proof. Using the techniques developed in [16], one shows easily that

$$\mathrm{Hom}\,(P, \mathrm{R}\Gamma(U, \mathrm{IB}(\mathcal{C}_{\infty,X}^{(p,q)}))) \simeq \mathrm{R}\Gamma(U, h_P(\mathrm{IB}(\mathcal{C}_{\infty,X}^{(p,q)})))$$

for any projective object P of $\mathcal{I}nd(\mathcal{B}an)$. Therefore, the result will be true if the sheaf of abelian groups $h_P(\mathrm{IB}(\mathcal{C}_{\infty,X}^{(p,q)}))$ is soft. This follows from the fact that it has clearly a canonical structure of $\mathcal{C}_{\infty,X}$-module. $\qquad\square$

Theorem 3.5. *If U is an open subset of X such that*

$$H^k(U, \mathcal{O}_X) \simeq 0 \qquad (k > 0)$$

algebraically, then

$$\mathrm{R}\Gamma(U, \mathrm{IB}(\mathcal{O}_X)) \simeq \mathrm{IB}(\mathcal{O}_X(U)).$$

Proof. As is well-known, since $\mathcal{C}_{\infty,U}^{(p,q)}$ is a soft sheaf, the Dolbeault complex

$$0 \to \mathcal{C}_{\infty,X}^{(0,0)} \xrightarrow{\overline{\partial}} \mathcal{C}_{\infty,X}^{(0,1)} \cdots \xrightarrow{\overline{\partial}} \mathcal{C}_{\infty,X}^{(0,n)} \to 0$$

is a $\Gamma(U, \cdot)$-acyclic resolution of $\mathcal{O}_X$. Therefore, $\mathrm{R}\Gamma(U, \mathcal{O}_X)$ is given by the complex

$$0 \to \Gamma(U, \mathcal{C}_{\infty,X}^{(0,0)}) \xrightarrow{\overline{\partial}} \Gamma(U, \mathcal{C}_{\infty,X}^{(0,1)}) \cdots \xrightarrow{\overline{\partial}} \Gamma(U, \mathcal{C}_{\infty,X}^{(0,n)}) \to 0.$$

Moreover, since $H^k(U, \mathcal{O}_X) \simeq 0$ for $k > 0$, the sequence

$$0 \to \Gamma(U, \mathcal{O}_X) \to \Gamma(U, \mathcal{C}_{\infty,X}^{(0,0)}) \xrightarrow{\overline{\partial}} \Gamma(U, \mathcal{C}_{\infty,X}^{(0,1)}) \cdots \xrightarrow{\overline{\partial}} \Gamma(U, \mathcal{C}_{\infty,X}^{(0,n)}) \to 0$$

is algebraically exact. Since $\mathcal{O}_X(U)$ and $\mathcal{C}_{\infty,X}^{(p,q)}(U)$ are FN spaces, the last sequence is strictly exact in $\mathcal{T}c$. Using [16, Proposition 3.2.26], one sees easily that the sequence

$$0 \to \Gamma(U, \mathrm{IB}(\mathcal{O}_X)) \to \Gamma(U, \mathrm{IB}(\mathcal{C}_{\infty,X}^{(0,0)})) \cdots \to \Gamma(U, \mathrm{IB}(\mathcal{C}_{\infty,X}^{(0,n)})) \to 0 \qquad (*)$$

is strictly exact in $\mathcal{I}nd(\mathcal{B}an)$. For any open ball b of X, Cartan's Theorem B shows that

$$H^k(b, \mathcal{O}_X) \simeq 0 \qquad (k > 0).$$

Hence, the sequence

$$0 \to \Gamma(b, \mathrm{IB}(\mathcal{O}_X)) \to \Gamma(b, \mathrm{IB}(\mathcal{C}_{\infty,X}^{(0,0)})) \cdots \to \Gamma(b, \mathrm{IB}(\mathcal{C}_{\infty,X}^{(0,n)})) \to 0$$

is strictly exact in $\mathcal{I}nd(\mathcal{B}an)$. Filtering inductive limits being exact in $\mathcal{I}nd(\mathcal{B}an)$, we see that

$$0 \to \mathrm{IB}(\mathcal{O}_X) \to \mathrm{IB}(\mathcal{C}_{\infty,X}^{(0,0)}) \to \mathrm{IB}(\mathcal{C}_{\infty,X}^{(0,1)}) \cdots \to \mathrm{IB}(\mathcal{C}_{\infty,X}^{(0,n)}) \to 0$$

is a strictly exact sequence of sheaves with values in $\mathcal{I}nd(\mathcal{B}an)$. Moreover, since, by Proposition 3.4, $\mathrm{IB}(\mathcal{C}_{\infty,U}^{(p,q)})$ is $\Gamma(U, \cdot)$-acyclic, $\mathrm{R}\Gamma(U, \mathrm{IB}(\mathcal{O}_X))$ is given by

$$0 \to \Gamma(U, \mathrm{IB}(\mathcal{C}_{\infty,X}^{(0,0)})) \to \Gamma(U, \mathrm{IB}(\mathcal{C}_{\infty,X}^{(0,1)})) \cdots \to \Gamma(U, \mathrm{IB}(\mathcal{C}_{\infty,X}^{(0,n)})) \to 0.$$

The sequence (*) being strictly exact, we get

$$\mathrm{R}\Gamma(U, \mathrm{IB}(\mathcal{O}_X)) \simeq \Gamma(U, \mathrm{IB}(\mathcal{O}_X)).$$

$\square$

Proposition 3.6. *If X is a Stein manifold and K is a holomorphically convex compact subset of X, we have*

$$R\Gamma(K, \mathrm{IB}(\mathcal{O}_X)) \simeq \mathrm{IB}(\mathcal{O}_X(K)).$$

Proof. It is well-known that K has a fundamental system $\mathcal{V}$ of Stein open neighborhoods. By tautness, it follows that for $k > 0$, we have

$$LH^k(K, \mathrm{IB}(\mathcal{O}_X)) \simeq \varinjlim_{V \in \mathcal{V}} LH^k(V, \mathrm{IB}(\mathcal{O}_X)) \simeq 0$$

where the second isomorphism follows from Theorem 3.5. Hence, using Proposition 3.3, we get

$$R\Gamma(K, \mathrm{IB}(\mathcal{O}_X)) \simeq \Gamma(K, \mathrm{IB}(\mathcal{O}_X)) \simeq \mathrm{IB}(\mathcal{O}_X(K)).$$

$\square$

Remark 3.7. Note that all the results in this section clearly hold if we replace $\mathcal{O}_X$ by the sheaf of holomorphic sections of holomorphic vector bundle. In particular, they hold for the sheaf Ω_X^p of holomorphic p-forms.

4   A factorization formula for $\mathrm{IB}(\mathcal{O}_{X \times Y})$

Definition 4.1. For any $\rho = (\rho_1, \cdots, \rho_p) \in {]0, +\infty[}^p$, we set

$$\Delta_\rho = \{z \in \mathbb{C}^p : |z_1| < \rho_1, \cdots, |z_p| < \rho_p\}$$

and we denote by A_ρ the object of $\mathcal{T}c$ defined by endowing

$$A_\rho = \{(a_\alpha)_{\alpha \in \mathbb{N}^p} : \sum_\alpha |a_\alpha| \rho^\alpha < +\infty\}$$

with the norm

$$\|(a_\alpha)_{\alpha \in \mathbb{N}^p}\| = \sum_\alpha |a_\alpha| \rho^\alpha.$$

Lemma 4.2. *For any $\rho \in {]0, +\infty[}^p$, we have the isomorphism*

$$A_\rho \simeq l^1(\mathbb{N}^p).$$

In particular, A_ρ is a Banach space.

Proof. This follows directly from the fact that the application

$$u : A_\rho \to l^1(\mathbb{N}^p)$$

defined by $u((a_\alpha)_{\alpha \in \mathbb{N}^p}) = (a_\alpha \rho^\alpha)_{\alpha \in \mathbb{N}^p}$ is continuous and bijective. $\qquad\square$

Lemma 4.3. *For any $p \in \mathbb{N}$,*

$$\varinjlim_{\rho \in]0,+\infty[^p} \mathrm{IB}(\mathcal{O}_{\mathbb{C}^p}(\Delta_\rho)) \simeq \varinjlim_{\rho \in]0,+\infty[^p} \mathrm{IB}(A_\rho).$$

Proof. This follows directly from the fact that the canonical restriction morphism

$$\mathcal{O}_{\mathbb{C}^p}(\Delta_{\rho'}) \to \mathcal{O}_{\mathbb{C}^p}(\Delta_\rho)$$

may be factored through A_ρ for $\rho' > \rho$. $\qquad\square$

Proposition 4.4. *Assume X, Y are complex analytic manifolds. Then, there is a canonical isomorphism*

$$\mathrm{IB}(\mathcal{O}_X) \hat{\boxtimes}^L \mathrm{IB}(\mathcal{O}_Y) \simeq \mathrm{IB}(\mathcal{O}_{X \times Y}).$$

Proof. Let U, V be open subsets of X and Y. The map

$$u_{U,V} : \mathcal{O}_X(U) \times \mathcal{O}_Y(V) \to \mathcal{O}_{X \times Y}(U \times V)$$

defined by setting

$$u_{U,V}(f,g)(u,v) = f(u)g(v)$$

is clearly bilinear and continuous. Hence, it induces a morphism

$$\mathcal{O}_X(U) \otimes_\pi \mathcal{O}_Y(V) \to \mathcal{O}_{X \times Y}(U \times V)$$

and by Proposition 1.15, we get a morphism

$$\mu_{U,V} : \mathrm{IB}(\mathcal{O}_X(U)) \hat{\otimes} \mathrm{IB}(\mathcal{O}_Y(V)) \to \mathrm{IB}(\mathcal{O}_{X \times Y}(U \times V))$$

which is clearly well-behaved with respect to the restriction of U or V. Therefore, we get a canonical morphism

$$\mu : \mathrm{IB}(\mathcal{O}_X) \hat{\boxtimes} \mathrm{IB}(\mathcal{O}_Y) \to \mathrm{IB}(\mathcal{O}_{X \times Y}).$$

To show that it is an isomorphism, it is sufficient to work at the level of germs and to prove that

$$\mu_{(x,y)} : \mathrm{IB}(\mathcal{O}_X)_x \hat{\otimes} \mathrm{IB}(\mathcal{O}_Y)_y \to \mathrm{IB}(\mathcal{O}_{X \times Y})_{(x,y)}$$

is an isomorphism. The problem being local, we may assume $X = \mathbb{C}^p$, $Y = \mathbb{C}^{p'}$, $x = 0$, $y = 0$. In this case, Lemma 4.3 shows that

$$\mathrm{IB}(\mathcal{O}_X)_x \simeq \varinjlim_{\rho \in]0,+\infty[^p} \mathrm{IB}(A_\rho), \qquad \mathrm{IB}(\mathcal{O}_Y)_y \simeq \varinjlim_{\rho' \in]0,+\infty[^{p'}} \mathrm{IB}(A_{\rho'})$$

and

$$\mathrm{IB}(\mathcal{O}_{X \times Y})_{(x,y)} \simeq \varinjlim_{(\rho,\rho') \in]0,+\infty[^{p+p'}} \mathrm{IB}(A_{(\rho,\rho')}).$$

A direct computation shows that through these isomorphisms $\mu_{x,y}$ corresponds to the inductive limit of the maps

$$\tau_{\rho,\rho'} : \mathrm{IB}(A_\rho) \,\hat{\otimes}\, \mathrm{IB}(A_{\rho'}) \longrightarrow \mathrm{IB}(A_{(\rho,\rho')})$$

associated to the continuous bilinear maps

$$t_{\rho,\rho'} : A_\rho \times A_{\rho'} \longrightarrow A_{(\rho,\rho')}$$

defined by

$$t_{\rho,\rho'}((a_\alpha)_{\alpha \in \mathbb{N}^p}, (a'_{\alpha'})_{\alpha' \in \mathbb{N}^{p'}}) = (a_\alpha a'_{\alpha'})_{(\alpha,\alpha') \in \mathbb{N}^{p+p'}}.$$

Since the diagram

$$
\begin{array}{ccc}
\mathrm{IB}(A_\rho) \,\hat{\otimes}\, \mathrm{IB}(A_{\rho'}) & \xrightarrow{\ \sim\ } & \text{``}A_\rho \,\hat{\otimes}\, A_{\rho'}\text{''} \\
{\scriptstyle \tau_{\rho,\rho'}}\Big\downarrow & & \Big\downarrow{\scriptstyle \text{``}t_{\rho,\rho'}\text{''}} \\
\mathrm{IB}(A_{(\rho,\rho')}) & \xrightarrow{\ \sim\ } & \text{``}A_{(\rho,\rho')}\text{''}
\end{array}
$$

is clearly commutative, to prove that $\mu_{(x,y)}$ is an isomorphism, it is sufficient to prove that $t_{\rho,\rho'}$ is an isomorphism. Thanks to Lemma 4.2, this fact is an easy consequence of the well-known isomorphism

$$l^1(\mathbb{N}^p) \,\hat{\otimes}\, l^1(\mathbb{N}^{p'}) \simeq l^1(\mathbb{N}^{p+p'}).$$

By Proposition 3.3,

$$\mathrm{IB}(\mathcal{O}_X)_x \simeq \Gamma(\{x\}, \mathrm{IB}(\mathcal{O}_X)) \simeq \mathrm{IB}(\mathcal{O}_X(\{x\})).$$

Since $\mathcal{O}_X(\{x\})$ is a DFN space, Proposition 2.19, shows that

$$\mathrm{IB}(\mathcal{O}_X)_x \,\hat{\otimes}^L\, \mathrm{IB}(\mathcal{O}_Y)_y \simeq \mathrm{IB}(\mathcal{O}_X)_x \,\hat{\otimes}\, \mathrm{IB}(\mathcal{O}_Y)_y.$$

Therefore,

$$\mathrm{IB}(\mathcal{O}_X) \hat{\boxtimes}^L \mathrm{IB}(\mathcal{O}_Y) \simeq \mathrm{IB}(\mathcal{O}_X) \hat{\boxtimes} \mathrm{IB}(\mathcal{O}_Y) \simeq \mathrm{IB}(\mathcal{O}_{X \times Y})$$

as requested.   □

Corollary 4.5. *If A, B are subsets of X and Y then*

$$\mathrm{R}\Gamma_c(A \times B, \mathrm{IB}(\mathcal{O}_{X \times Y})) \simeq \mathrm{R}\Gamma_c(A, \mathrm{IB}(\mathcal{O}_X)) \hat{\otimes}^L \mathrm{R}\Gamma_c(B; \mathrm{IB}(\mathcal{O}_Y)).$$

In particular, if X, Y are Stein manifolds and K, L are holomorphically convex compact subsets of X and Y, then

$$\mathrm{IB}(\mathcal{O}_{X \times Y}(K \times L)) \simeq \mathrm{IB}(\mathcal{O}_X(K)) \hat{\otimes} \mathrm{IB}(\mathcal{O}_Y(L)).$$

Proof. The first part is a direct consequence of Theorem 4.4 and the Künneth theorem for sheaves with values in $\mathcal{I}nd(\mathcal{B}an)$. The second part follows from the first using Proposition 3.6, Proposition 2.19 and the fact that $\mathcal{O}_X(K)$ is a DFN space. $\qquad \square$

5 Poincaré duality for $\mathrm{IB}(\mathcal{O}_X)$

Proposition 5.1. *Assume X, Y are complex analytic manifolds of dimension d_X and d_Y. Then, there is a canonical integration morphism*

$$\int_X : Rq_{Y!}(\mathrm{IB}(\Omega_{X \times Y})[d_{X \times Y}]) \longrightarrow \mathrm{IB}(\Omega_Y)[d_Y].$$

Proof. Recall that integration along the fibers of q_Y (i.e. on X) defines morphisms

$$\int_X : q_{Y!}(\mathcal{C}_{\infty,X \times Y}^{p+d_X,q+d_X}) \longrightarrow \mathcal{C}_{\infty,Y}^{p,q} \qquad (p, q \in \mathbb{Z}) \tag{$*$}$$

which are compatible with ∂ and $\overline{\partial}$. Fix p, $q \in \mathbb{Z}$. Let K be a compact subset of X and let U be an open subset of Y. One checks easily that the morphism

$$\int_X : \Gamma_{K \times U}(X \times U; \mathcal{C}_{\infty,X \times Y}^{p+d_X,q+d_X}) \longrightarrow \Gamma(U; \mathcal{C}_{\infty,Y}^{p,q})$$

is continuous for the canonical topologies. Applying IB, we get a morphism

$$\Gamma_{K \times U}(X \times U; \mathrm{IB}(\mathcal{C}_{\infty,X \times Y}^{p+d_X,q+d_X})) \longrightarrow \Gamma(U; \mathrm{IB}(\mathcal{C}_{\infty,Y}^{p,q})).$$

Taking the inductive limit on K, we get a morphism

$$\Gamma(U; q_{Y!}(\mathrm{IB}(\mathcal{C}_{\infty,X \times Y}^{p+d_X,q+d_X}))) \longrightarrow \Gamma(U; \mathrm{IB}(\mathcal{C}_{\infty,Y}^{p,q}))$$

and hence a morphism

$$q_{Y!}(\mathrm{IB}(\mathcal{C}_{\infty,X \times Y}^{p+d_X,q+d_X})) \longrightarrow \mathrm{IB}(\mathcal{C}_{\infty,Y}^{p,q})$$

of sheaves with values in $\mathcal{I}nd(\mathcal{B}an)$. Thanks to the compatibility of (*) with ∂ and $\overline{\partial}$, we also get a morphism of complexes

$$q_{Y!}(\mathrm{IB}(\mathcal{C}_{\infty,}^{d_{X\times Y},\cdot})[d_{X\times Y}]) \longrightarrow \mathrm{IB}(\mathcal{C}_{\infty,Y}^{d_X,\cdot})[d_Y].$$

Using the properties of Dolbeault resolutions, we get the requested integration morphism

$$\int_X : Rq_{Y!}(\mathrm{IB}(\Omega_{X\times Y})[d_{X\times Y}]) \longrightarrow \mathrm{IB}(\Omega_Y)[d_Y].$$

$\square$

Remark 5.2. Assume X, Y, Z are complex analytic manifolds. Then, one checks easily that Fubini Theorem gives rise to the commutative diagram

$$
\begin{array}{ccc}
Rq_{Z!}(Rq_{Y\times Z!}(\mathrm{IB}(\Omega_{X\times Y\times Z})[d_{X\times Y\times Z}])) & \xrightarrow{\int_X} & Rq_{Z!}(\mathrm{IB}(\Omega_{Y\times Z})[d_{Y\times Z}]) \\
\downarrow{\wr} & & \downarrow{\int_Y} \\
Rq_{Z!}(\mathrm{IB}(\Omega_{X\times Y\times Z})[d_{X\times Y\times Z}]) & \xrightarrow[\int_{X\times Y}]{} & \mathrm{IB}(\Omega_Z)[d_Z]
\end{array}
$$

Moreover, using the linearity of the integral, one gets the commutative diagram

$$
\begin{array}{ccc}
Rq_{Y!}(\mathrm{IB}(\Omega_{X\times Y})[d_{X\times Y}]) \,\hat{\otimes}\, \mathrm{IB}(\mathcal{O}_Y) & \xrightarrow{\int_X \hat{\otimes}\,\mathrm{id}} & \mathrm{IB}(\Omega_Y)[d_Y] \,\hat{\otimes}\, \mathrm{IB}(\mathcal{O}_Y) \\
{\scriptstyle\text{projection}}\downarrow & & \downarrow{\scriptstyle\text{cup}} \\
Rq_{Y!}(\mathrm{IB}(\Omega_{X\times Y})[d_{X\times Y}] \,\hat{\otimes}\, q_Y^{-1}\,\mathrm{IB}(\mathcal{O}_Y)) & & \\
{\scriptstyle\text{cup}}\downarrow & & \\
Rq_{Y!}(\mathrm{IB}(\Omega_{X\times Y})[d_{X\times Y}]) & \xrightarrow[\int_X]{} & \mathrm{IB}(\Omega_Y)[d_Y]
\end{array}
$$

Theorem 5.3. *Assume X is a complex analytic manifold of dimension d_X and denote $a_X : X \to \{\mathrm{pt}\}$ the canonical map. Then, the morphism*

$$\mathrm{IB}(\Omega_X^{d_X-p})[d_X] \to \mathrm{D}(\mathrm{IB}(\Omega_X^p)).$$

induced by adjunction from

$$\int_X \circ \smile : a_{X!}(\mathrm{IB}(\Omega_X^{d_X-p}[d_X]) \,\hat{\otimes}\, \mathrm{IB}(\Omega_X^p)) \to \mathrm{IB}(\mathbb{C})$$

is an isomorphism.

Proof. The problem being local, it is sufficient to treat the case $p = 0$ and to show that the morphism

$$\mathrm{R}\Gamma(U; \mathrm{IB}(\Omega_U)[d_U]) \longrightarrow \mathrm{RL}\left(\mathrm{R}\Gamma_c(U; \mathrm{IB}(\mathcal{O}_U)), \mathrm{IB}(\mathbb{C})\right)$$

obtained by adjunction from

$$\int_X \circ \smile: \mathrm{R}\Gamma(U; \mathrm{IB}(\Omega_U)[d_U]) \,\hat{\otimes}^L\, \mathrm{R}\Gamma_c(U; \mathrm{IB}(\mathcal{O}_U)) \longrightarrow \mathrm{IB}(\mathbb{C})$$

is an isomorphism for any open interval U of $\mathbb{C}_{d_U}$. This follows directly from Proposition 5.5 below with V reduced to a point. $\qquad\square$

Remark 5.4. As we will show elsewhere, the preceding theorem may be used to simplify the topological duality theory for coherent analytic sheaves.

Proposition 5.5. *Assume U is an open interval of $\mathbb{C}^{d_U}$ and V is an open interval of $\mathbb{C}^{d_V}$. Then, the canonical morphism*

$$\varphi_{U,V} : \mathrm{R}\Gamma(U \times V, \mathrm{IB}(\Omega_{U \times V})[d_{U \times V}]) \longrightarrow \mathrm{RL}\left(\mathrm{R}\Gamma_c(U; \mathrm{IB}(\mathcal{O}_U)), \mathrm{R}\Gamma(V; \mathrm{IB}(\Omega_V)[d_V])\right)$$

obtained by adjunction from

$$\int_X \circ \smile: \mathrm{R}\Gamma(U \times V, \mathrm{IB}(\Omega_{U \times V})[d_{U \times V}]) \,\hat{\otimes}^L\, \mathrm{R}\Gamma_c(U; \mathrm{IB}(\mathcal{O}_U)) \longrightarrow \mathrm{R}\Gamma(V; \mathrm{IB}(\Omega_V)[d_V])$$

is an isomorphism.

Proof. Let W be an open interval of $\mathbb{C}^{d_W}$ and assume that $\varphi_{U,V \times W}$ and $\varphi_{V,W}$ are isomorphisms. Then, we have successively

$$\mathrm{R}\Gamma(U \times V \times W; \mathrm{IB}(\Omega_{U \times V \times W})[d_{U \times V \times W}])$$
$$\simeq \mathrm{RL}\left(\mathrm{R}\Gamma_c(U; \mathrm{IB}(\mathcal{O}_U)), \mathrm{R}\Gamma(V \times W, \mathrm{IB}(\Omega_{V \times W})[d_{V \times W}])\right) \qquad (1)$$
$$\simeq \mathrm{RL}\left(\mathrm{R}\Gamma_c(U; \mathrm{IB}(\mathcal{O}_U)), \mathrm{RL}\left(\mathrm{R}\Gamma_c(V; \mathrm{IB}(\mathcal{O}_V)), \mathrm{R}\Gamma(W; \mathrm{IB}(\Omega_W)[d_W])\right)\right) \qquad (2)$$
$$\simeq \mathrm{RL}\left(\mathrm{R}\Gamma_c(U; \mathrm{IB}(\mathcal{O}_U)) \,\hat{\otimes}^L\, \mathrm{R}\Gamma_c(V; \mathrm{IB}(\mathcal{O}_V)), \mathrm{R}\Gamma(W; \mathrm{IB}(\Omega_W)[d_W])\right) \qquad (3)$$
$$\simeq \mathrm{RL}\left(\mathrm{R}\Gamma_c(U \times V; \mathrm{IB}(\mathcal{O}_{U \times V})), \mathrm{R}\Gamma(W; \mathrm{IB}(\Omega_W)[d_W])\right), \qquad (4)$$

where (1) and (2) follow from our assumptions, (3) is obtained by adjunction and (4) comes from Corollary 4.5. Using Remark 5.2, we check easily that the composition of the preceding isomorphisms is equal to $\varphi_{U \times V, W}$. Hence, an induction on d_U reduces the problem to the case where $d_U = 1$. This will be dealt with in Proposition 5.6 below. $\qquad\square$

Proposition 5.6. *Assume U is an open interval of $\mathbb{C}$ and V is an open interval of $\mathbb{C}^n$. Then, the canonical morphism*

$$\mathrm{R\Gamma}(U \times V, \mathrm{IB}(\Omega_{U \times V})[d_{U \times V}]) \longrightarrow \mathrm{RL}\left(\mathrm{R\Gamma}_c(U; \mathrm{IB}(\mathcal{O}_U)), \mathrm{R\Gamma}(V; \mathrm{IB}(\Omega_V)[d_V])\right)$$

is an isomorphism.

Proof. For $P = \overline{U}$, sheaf theory gives us the two distinguished triangles

$$\mathrm{R\Gamma}_{\partial P \times V}(\mathbb{C} \times V, \mathrm{IB}(\Omega_{\mathbb{C} \times V})) \to \mathrm{R\Gamma}_{P \times V}(\mathbb{C} \times V, \mathrm{IB}(\Omega_{\mathbb{C} \times V})) \to \mathrm{R\Gamma}(U \times V, \mathrm{IB}(\Omega_{U \times V})) \xrightarrow{+1}$$

and

$$\mathrm{R\Gamma}_c(U, \mathrm{IB}(\mathcal{O}_U)) \to \mathrm{R\Gamma}(P, \mathrm{IB}(\mathcal{O}_\mathbb{C})) \to \mathrm{R\Gamma}(\partial P, \mathrm{IB}(\mathcal{O}_\mathbb{C})) \xrightarrow{+1} .$$

If we apply the functor $\mathrm{RL}\left(\cdot, \mathrm{IB}(\Omega_V(V))\right)$ to the last triangle, we obtain the morphism of distinguished triangles

$$
\begin{array}{ccc}
\mathrm{R\Gamma}_{\partial P \times V}(\mathbb{C} \times V, \mathrm{IB}(\mathcal{O}_{\mathbb{C} \times V}))[1] & \xrightarrow{\ \alpha\ } & \mathrm{RL}\left(\mathrm{R\Gamma}(\partial P, \mathrm{IB}(\mathcal{O}_\mathbb{C})), \mathrm{IB}(\Omega_V(V))\right) \\
\downarrow & & \downarrow \\
\mathrm{R\Gamma}_{P \times V}(\mathbb{C} \times V, \mathrm{IB}(\mathcal{O}_{\mathbb{C} \times V}))[1] & \xrightarrow{\ \beta\ } & \mathrm{RL}\left(\mathrm{R\Gamma}(P, \mathrm{IB}(\mathcal{O}_\mathbb{C})), \mathrm{IB}(\Omega_V(V))\right) \\
\downarrow & & \downarrow \\
\mathrm{R\Gamma}(U \times V, \mathrm{IB}(\mathcal{O}_{U \times V}))[1] & \xrightarrow{\ \gamma\ } & \mathrm{RL}\left(\mathrm{R\Gamma}_c(U, \mathrm{IB}(\mathcal{O}_\mathbb{C})), \mathrm{IB}(\Omega_V(V))\right) \\
\downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1} \\
\end{array}
$$

where α and β are isomorphisms of the type considered in Proposition 5.7 below (∂P is a finite union of closed intervals of $\mathbb{C}$). It follows that γ is an isomorphism. $\qquad\square$

Proposition 5.7. *Assume K is a finite union of closed intervals of $\mathbb{C}$ and V is an open interval of $\mathbb{C}^n$. Then, the canonical morphism*

$$\mathrm{R\Gamma}_{K \times V}(\mathbb{C} \times V; \mathrm{IB}(\Omega_{\mathbb{C} \times V})[d_{\mathbb{C} \times V}]) \longrightarrow \mathrm{RL}\left(\mathrm{R\Gamma}(K; \mathrm{IB}(\mathcal{O}_\mathbb{C})), \mathrm{R\Gamma}(V; \mathrm{IB}(\Omega_V)[d_V])\right)$$

obtained by adjunction from

$$\int_\mathbb{C} \circ \smile : \mathrm{R\Gamma}_{K \times V}(\mathbb{C} \times V; \mathrm{IB}(\Omega_{\mathbb{C} \times V})[d_{\mathbb{C} \times V}]) \,\hat{\otimes}^L\, \mathrm{R\Gamma}(K; \mathrm{IB}(\mathcal{O}_\mathbb{C})) \longrightarrow \mathrm{R\Gamma}(V; \mathrm{IB}(\Omega_V)[d_V])$$

is an isomorphism.

Proof. Assume first that K is a closed interval of $\mathbb{C}$. Since $P \times V$ is closed in $\mathbb{C} \times V$, we have the distinguished triangle

$$\mathrm{R\Gamma}_{P \times V}(\mathbb{C} \times V, \mathrm{IB}(\mathcal{O}_{\mathbb{C} \times V})) \to \mathrm{R\Gamma}(\mathbb{C} \times V, \mathrm{IB}(\mathcal{O}_{\mathbb{C} \times V})) \to \mathrm{R\Gamma}((\mathbb{C} \setminus P) \times V, \mathrm{IB}(\mathcal{O}_{\mathbb{C} \times V})) \xrightarrow{+1}$$

By Cartan's Theorem B and Theorem 3.5, we have the isomorphisms

$$\mathrm{R\Gamma}(\mathbb{C} \times V, \mathrm{IB}(\mathcal{O}_{\mathbb{C} \times V})) \simeq \mathrm{IB}(\mathcal{O}_{\mathbb{C} \times V}(\mathbb{C} \times V))$$

and

$$\mathrm{R\Gamma}((\mathbb{C} \setminus P) \times V, \mathrm{IB}(\mathcal{O}_{\mathbb{C} \times V})) \simeq \mathrm{IB}(\mathcal{O}_{\mathbb{C} \times V}((\mathbb{C} \setminus P) \times V)).$$

Hence, the long exact sequence associated to the preceding distinguished triangle ensures that

$$LH^k_{P \times V}(\mathbb{C} \times V, \mathrm{IB}(\mathcal{O}_{\mathbb{C} \times V})) = 0 \qquad \forall k \geq 2$$

and that the sequence

$$0 \longrightarrow LH^0_{P \times V}(\mathbb{C} \times V, \mathrm{IB}(\mathcal{O}_{\mathbb{C} \times V})) \longrightarrow \mathrm{IB}(\mathcal{O}_{\mathbb{C} \times V}(\mathbb{C} \times V)) \longrightarrow$$
$$\longrightarrow IB(\mathcal{O}_{\mathbb{C} \times V}((\mathbb{C} \setminus P) \times V)) \longrightarrow LH^1_{P \times V}(\mathbb{C} \times V, \mathrm{IB}(\mathcal{O}_{\mathbb{C} \times V})) \longrightarrow 0$$

is strictly exact. Applying the functor IB to the sequence of Proposition 5.9 below, we get the split exact sequence

$$0 \to \mathrm{IB}(\mathcal{O}_{\mathbb{C} \times V}(\mathbb{C} \times V)) \to \mathrm{IB}(\mathcal{O}_{\mathbb{C} \times V}((\mathbb{C} \setminus P) \times V)) \to \mathrm{IB}(\mathrm{L}_b(\mathcal{O}_{\mathbb{C}}(P), \mathcal{O}_V(V))) \to 0 \quad (*)$$

in $\mathcal{I}nd(\mathcal{B}an)$. Therefore,

$$LH^0_{P \times V}(\mathbb{C} \times V, \mathrm{IB}(\mathcal{O}_{\mathbb{C} \times V})) = 0$$

and

$$LH^1_{P \times V}(\mathbb{C} \times V, \mathrm{IB}(\mathcal{O}_{\mathbb{C} \times V})) \simeq \mathrm{IB}(\mathrm{L}_b(\mathcal{O}_{\mathbb{C}}(P), \mathcal{O}_V(V))).$$

Combining these results with Proposition 1.13, Theorem 2.13, Theorem 3.5 and Proposition 3.6, we obtain successively

$$\mathrm{R\Gamma}_{P \times V}(\mathbb{C} \times V, \mathrm{IB}(\mathcal{O}_{\mathbb{C} \times V})) \simeq \mathrm{L}\,(\mathrm{IB}(\mathcal{O}_{\mathbb{C}}(P)), \mathrm{IB}(\mathcal{O}_V(V)))[-1]$$
$$\simeq \mathrm{RL}\,(\mathrm{R\Gamma}(P; \mathrm{IB}(\mathcal{O}_{\mathbb{C}})), \mathrm{R\Gamma}(V; \mathrm{IB}(\mathcal{O}_V)))[-1].$$

Thanks to Proposition 5.8 below, it follows easily that the canonical morphism

$$\mathrm{R\Gamma}_{P \times V}(\mathbb{C} \times V; \mathrm{IB}(\Omega_{\mathbb{C} \times V})[d_{\mathbb{C} \times V}]) \to \mathrm{RL}\,(\mathrm{R\Gamma}(P; \mathrm{IB}(\mathcal{O}_{\mathbb{C}})), \mathrm{R\Gamma}(V; \mathrm{IB}(\Omega_V)[d_V]))$$

is an isomorphism.

Assume now that the result has been established when K is a union of $k < N$ closed intervals of $\mathbb{C}$ and let us prove it when

$$K = \bigcup_{i=1}^{N} P_i$$

where P_i $(i = 1, \cdots, N)$ is a closed interval of $\mathbb{C}$. Set $L = \bigcup_{i=1}^{N-1} P_i$ and $Q = P_N$. By the Mayer-Vietoris theorem associated to the decomposition $K = L \cup Q$, we have the distinguished triangle

$$\mathrm{R}\Gamma(K, \mathrm{IB}(\mathcal{O}_{\mathbb{C}}))$$
$$\downarrow$$
$$\mathrm{R}\Gamma(L, \mathrm{IB}(\mathcal{O}_{\mathbb{C}})) \oplus \mathrm{R}\Gamma(Q, \mathrm{IB}(\mathcal{O}_{\mathbb{C}}))$$
$$\downarrow$$
$$\mathrm{R}\Gamma(L \cap Q, \mathrm{IB}(\mathcal{O}_{\mathbb{C}}))$$
$$\downarrow{\scriptstyle +1}$$

Applying the functor $\mathrm{RL}\,(\cdot, \mathrm{IB}(\Omega_V(V)))$, we obtain the distinguished triangle

$$A = \mathrm{RL}\,(\mathrm{R}\Gamma(L \cap Q, \mathrm{IB}(\mathcal{O}_{\mathbb{C}})), \mathrm{IB}(\Omega_V(V)))$$
$$\downarrow$$
$$B = \mathrm{RL}\,(\mathrm{R}\Gamma(L, \mathrm{IB}(\mathcal{O}_{\mathbb{C}})) \oplus \mathrm{R}\Gamma(Q, \mathrm{IB}(\mathcal{O}_{\mathbb{C}})), \mathrm{IB}(\Omega_V(V)))$$
$$\downarrow$$
$$C = \mathrm{RL}\,(\mathrm{R}\Gamma(K, \mathrm{IB}(\mathcal{O}_{\mathbb{C}})), \mathrm{IB}(\Omega_V(V)))$$
$$\downarrow{\scriptstyle +1}$$

Now, consider the Mayer-Vietoris distinguished triangle

$$A' = \mathrm{R}\Gamma_{(L \cap Q) \times V}(\mathbb{C} \times V, \mathrm{IB}(\Omega_{\mathbb{C} \times V}))$$
$$\downarrow$$
$$B' = \mathrm{R}\Gamma_{L \times V}(\mathbb{C} \times V, \mathrm{IB}(\Omega_{\mathbb{C} \times V})) \oplus \mathrm{R}\Gamma_{Q \times V}(\mathbb{C} \times V, \mathrm{IB}(\Omega_{\mathbb{C} \times V}))$$
$$\downarrow$$
$$C' = \mathrm{R}\Gamma_{K \times V}(\mathbb{C} \times V, \mathrm{IB}(\Omega_{\mathbb{C} \times V}))$$
$$\downarrow$$

Since $L \cap Q = \bigcup_{i=1}^{N-1}(P_i \cap P_N)$ is a union of $N - 1$ closed intervals of $\mathbb{C}$, the canonical morphisms

$$A'[1] \to A \qquad \text{and} \qquad B'[1] \to B$$

are isomorphisms. The canonical diagram

$$\begin{array}{ccccccc}
A'[1] & \longrightarrow & B'[1] & \longrightarrow & C'[1] & \xrightarrow{+1} & \\
\downarrow \wr & & \downarrow \wr & & \downarrow & & \\
A & \longrightarrow & B & \longrightarrow & C & \xrightarrow{+1} &
\end{array}$$

being commutative, the canonical morphism

$$C'[1] \to C$$

is also an isomorphism and the conclusion follows. $\square$

Proposition 5.8. *Let P be a compact interval of $\mathbb{C}$ and let V be an open interval of $\mathbb{C}^n$. Then,*

$$\int_{\mathbb{C}} : H^1_{P \times V}(\mathbb{C} \times V, \Omega_{\mathbb{C} \times V}) \to H^0(V, \Omega_V)$$

sends the class of

$$\omega = h(z,v)dz \wedge dv \in H^0(\mathbb{C} \setminus P, \Omega_{\mathbb{C} \times V})$$

to

$$\left(\int_{\partial P'} h(z,v)dz \right) dv$$

where P' is a compact interval of $\mathbb{C}$ such that $P'^\circ \supset P$.

Proof. Let $\mathcal{I}^\cdot$ be an injective resolution of $\Omega_{\mathbb{C} \times V}$. Denote

$$u^\cdot : \mathcal{C}^{(v+1,\cdot)}_{\infty, \mathbb{C} \times V} \to \mathcal{I}^\cdot$$

a morphism extending id $: \Omega_{\mathbb{C} \times V} \to \Omega_{\mathbb{C} \times V}$. The class c of ω in

$$H^1_{P \times V}(\mathbb{C} \times V, \Omega_{\mathbb{C} \times V}) \simeq H^1(\Gamma_P(\mathbb{C} \times V, \mathcal{I}^\cdot))$$

is represented by $d\sigma$ where $\sigma \in \Gamma(\mathbb{C} \times V, \mathcal{I}^0)$ extends $u^0(\omega) \in \Gamma((\mathbb{C} \setminus P) \times V, \mathcal{I}^0)$. Let φ be a function of class C_∞ on $\mathbb{C}$ equals to 1 on $\overline{\mathbb{C} \setminus P'}$ and to 0 on P'', P' and P'' being compact intervals such that $P''^\circ \supset P$, $P'^\circ \supset P''$. Then, it is clear that $u^0(\varphi\omega) \in \Gamma(\mathbb{C} \times V, \mathcal{I}^0)$ and that

$$\sigma - u^0(\varphi\omega) \in \Gamma_{c \times V}(\mathbb{C} \times V, \mathcal{I}^0).$$

Therefore, $d\sigma$ and $du^0(\varphi)$ give the same class in $H^1(\Gamma_{c \times V}(\mathbb{C} \times V, \mathcal{I}^{\cdot}))$. It follows that c' corresponds to the class of $\overline{\partial}(\varphi\omega)$ in $H^1(\Gamma_{c \times V}(\mathbb{C} \times V, \mathcal{C}^{(v+1,\cdot)}_{\infty,\mathbb{C} \times V}))$. Since c' represents the image of c by the canonical map

$$H^1_{P \times V}(\mathbb{C} \times V, \Omega_{\mathbb{C} \times V}) \longrightarrow H^1_{c \times V}(\mathbb{C} \times V, \Omega_{\mathbb{C} \times V}),$$

we see that

$$\int_{\mathbb{C}} c = \int_{\mathbb{C}} \overline{\partial}(\varphi\omega) = \int_{\overline{P' \setminus P''}} \overline{\partial}\varphi\omega = \int_{\partial P'} \varphi\omega - \int_{\partial P''} \varphi\omega = \int_{\partial P'} \omega.$$

Hence the conclusion. $\square$

Proposition 5.9. *Let P be a closed interval of $\mathbb{C}$ and let V be an open interval of $\mathbb{C}^n$. Then, in $\mathcal{T}c$, we have a split exact sequence of the form*

$$0 \longrightarrow \mathcal{O}_{\mathbb{C} \times V}(\mathbb{C} \times V) \xrightarrow{r} \mathcal{O}_{\mathbb{C} \times V}((\mathbb{C} \setminus P) \times V) \xrightarrow{T} \mathrm{L}_b(\mathcal{O}_{\mathbb{C}}(P), \mathcal{O}_V(V)) \longrightarrow 0.$$

where r is the canonical restriction map and T is defined by setting

$$T(h)(\varphi)(v) = \int_{\partial P'} h(z,v)g(z)dz$$

where g is a holomorphic extension of $\varphi \in \mathcal{O}_{\mathbb{C}}(P)$ on an open neighborhood U of P and P' is a compact interval of $\mathbb{C}$ such that $P'^{\circ} \supset P$ and $P' \subset U$.

Proof. Note that the definition of T is meaningful since the right hand side clearly does not depend on the choices of U, g and P'. It is also clear that the function $T(h)(\varphi)$ is holomorphic on V and that the operator T is linear. Let us show that T is continuous. Let p be a continuous semi-norm of $\mathrm{L}_b(\mathcal{O}_{\mathbb{C}}(P), \mathcal{O}_V(V))$. We may assume that there is a bounded subset B of $\mathcal{O}_{\mathbb{C}}(P)$ and a compact subset K of V such that

$$p(\tau) = \sup_{\varphi \in B} \sup_{v \in K} |\tau(\varphi)(v)|, \qquad \tau \in \mathrm{L}_b(\mathcal{O}_{\mathbb{C}}(P), \mathcal{O}_V(V)).$$

For $n > 0$, set $U_n = \{u \in \mathbb{C} : d(u, P) < 1/n\}$. By cofinality, we have

$$\mathcal{O}_{\mathbb{C}}(P) \simeq \varinjlim_{n > 0} \mathcal{O}_{\mathbb{C}}(U_n).$$

Moreover, for any $n > 0$, $\mathcal{O}_{\mathbb{C}}(U_n)$ is a Fréchet space and the restriction

$$\mathcal{O}_{\mathbb{C}}(U_n) \longrightarrow \mathcal{O}_{\mathbb{C}}(U_{n+1})$$

is injective. Hence, by [6, Chap. IV, § 19, 5.(5) (p. 225)], there is $n \in \mathbb{N}$ and a bounded subset B_n of $\mathcal{O}_\mathbb{C}(U_n)$ such that $B \subset r_{U_n}(B_n)$. Choosing a compact interval P'_n of $\mathbb{C}$ such that $P'^\circ_n \supset P$ and $P'_n \subset U_n$, we see that

$$p(T(h)) \leq \sup_{g \in B_n} \sup_{v \in K} \left| \int_{\partial P'_n} h(z, v) g(z) dz \right|$$

and we can find $C > 0$ such that

$$p(T(h)) \leq C \sup_{g \in B_n} \sup_{z \in \partial P'_n} |g(z)| \sup_{(z,v) \in \partial P'_n \times K} |h(z, v)|.$$

Let us consider the linear map

$$S : \mathrm{L}_b(\mathcal{O}_\mathbb{C}(P), \mathcal{O}_V(V)) \to \mathcal{O}_{\mathbb{C} \times V}((\mathbb{C} \setminus P) \times V)$$

defined by setting

$$S(\tau)(z, v) = \frac{1}{2i\pi} \tau \left(\frac{1}{z - u} \right)(v).$$

Let us check that S is continuous. Consider a compact subset K of $\mathbb{C} \setminus P$ and a compact subset L of V. The set

$$B_K = \left\{ \frac{1}{z - u} : z \in K \right\}$$

being bounded in $\mathcal{O}_\mathbb{C}(\mathbb{C} \setminus K)$, $r_{P, \mathbb{C} \setminus K}(B_K)$ is a bounded subset of $\mathcal{O}_\mathbb{C}(P)$ and we have

$$\sup_{(z,v) \in K \times L} |S(\tau)(z, v)| \leq \frac{1}{2\pi} \sup_{f \in r_{P, \mathbb{C} \setminus K}(B_K)} \sup_{v \in L} |\tau(f)(v)|.$$

For any $\tau \in \mathrm{L}_b(\mathcal{O}_\mathbb{C}(P), \mathcal{O}_V(V))$ and $\varphi \in \mathcal{O}_\mathbb{C}(P)$, there is an open U of $\mathbb{C}$, containing P and $g \in \mathcal{O}_\mathbb{C}(U)$ such that $\varphi = r_U(g)$. Let K be a closed interval included in U and such that $K^\circ \supset P$ and let $\mathcal{C}$ be the oriented boundary of K. For any $v \in V$, we have using the continuity of τ and Cauchy representation formula

$$\begin{aligned}
T(S(\tau))(\varphi)(v) &= \frac{1}{2i\pi} \int_\mathcal{C} \tau \left(\frac{1}{z - u} \right)(v) g(z) dz \\
&= \tau \left(\frac{1}{2i\pi} \int_\mathcal{C} \frac{g(z)}{z - u} dz \right)(v) \\
&= \tau(g)(v) = \tau(\varphi)(v).
\end{aligned}$$

It follows that $T \circ S = \mathrm{id}$ or, in other words, that S is a section of T.

Let us consider the continuous linear map

$$R : \mathcal{O}_{\mathbb{C} \times V}((\mathbb{C} \setminus P) \times V) \to \mathcal{O}_{\mathbb{C} \times V}(\mathbb{C} \times V)$$

defined as follows. Let $h \in \mathcal{O}_{\mathbb{C} \times V}((\mathbb{C} \setminus P) \times V)$ and $z \in \mathbb{C}$. Consider $R > 0$ such that

$$z \in P_R^\circ = \{z : d(z, P) < R\}.$$

Then, for any $v \in V$, we set

$$R(h)(z, v) = \frac{1}{2i\pi} \int_{\mathcal{C}_R} \frac{h(u, v)}{u - z} du$$

where $\mathcal{C}_R$ is the oriented boundary of P_R. Since, for any $f \in \mathcal{O}_{\mathbb{C} \times V}(\mathbb{C} \times V)$ and any $(z, v) \in \mathbb{C} \times V$, we have

$$R(r(f))(z, v) = \frac{1}{2i\pi} \int_{\mathcal{C}_R} \frac{r(f)(u, v)}{u - z} du = \frac{1}{2i\pi} \int_{\mathcal{C}_R} \frac{f(u, v)}{u - z} du = f(z, v),$$

we see that $R \circ r = \mathrm{id}$. The map R is thus a retraction of r.

Thanks to a well-known result of homological algebra, the proof will be complete if we show that

$$r \circ R + S \circ T = \mathrm{id}.$$

To this end, consider $h \in \mathcal{O}_{\mathbb{C} \times V}((\mathbb{C} \setminus P) \times V)$ and $(z, v) \in (\mathbb{C} \setminus P) \times V$. Fix $R > 0$ such that $z \in P_R^\circ$ and denote $\mathcal{C}_R$ the oriented boundary of P_R. Let $\mathcal{C}$ be the oriented boundary of a closed interval $K \subset P_R$ such that $z \notin K$ and $K^\circ \supset P$. Denoting Γ_R the oriented boundary of $P_R \setminus K^\circ$ and using Cauchy integral formula, we get

$$
\begin{aligned}
(r \circ R + S \circ T)(h)(z, v) &= \frac{1}{2i\pi} \int_{\mathcal{C}_R} \frac{h(\xi, v)}{\xi - z} d\xi + \frac{1}{2i\pi} T(h)\left(\frac{1}{z - u}\right)(v) \\
&= \frac{1}{2i\pi} \int_{\mathcal{C}_R} \frac{h(\xi, v)}{\xi - z} d\xi + \frac{1}{2i\pi} \int_{\mathcal{C}} \frac{h(\xi, v)}{z - \xi} d\xi \\
&= \frac{1}{2i\pi} \int_{\Gamma_R} \frac{h(\xi, v)}{\xi - z} d\xi \\
&= h(z, v).
\end{aligned}
$$

$\square$

Remark 5.10. The preceding result is a slightly more precise form of a special case of the Köthe-Grothendieck duality theorem (see [5] and [1, 2]).

6   A holomorphic Schwartz' kernel theorem

Definition 6.1. Let X and Y be complex analytic manifolds. We define $\Omega_{X \times Y}^{(r,s)}$ to be the subsheaf of $\Omega_{X \times Y}^{r+s}$ whose sections are the holomorphic differential forms that are locally a finite sum of forms of the type

$$\omega_{i,j} dx_{i_1} \wedge \cdots \wedge dx_{i_r} \wedge dy_{i_1} \wedge \cdots \wedge dy_{i_s}$$

where x and y are holomorphic local coordinate systems on X and Y.

Remark 6.2. Clearly, $\Gamma(W; \Omega_{X\times Y}^{(r,s)})$ has a canonical structure of FN space for any open subset W of $X \times Y$. Therefore, using Proposition 3.1, we see that $\mathrm{IB}(\Omega_{X\times Y}^{(r,s)})$ is a sheaf with value in $\mathcal{I}nd(\mathcal{B}an)$. Moreover, using Proposition 4.4, one can check easily that

$$\mathrm{IB}(\Omega_{X\times Y}^{(r,s)}) \simeq \mathrm{IB}(\Omega_X^r) \hat{\boxtimes}^L \mathrm{IB}(\Omega_Y^s).$$

Theorem 6.3. *Assume X, Y are complex analytic manifolds of dimension d_X, d_Y. Then, we have a canonical isomorphism*

$$\mathrm{IB}(\Omega_{X\times Y}^{(d_X-r,s)})[d_X] \simeq \mathrm{R}\mathcal{L}\,(q_X^{-1}\,\mathrm{IB}(\Omega_X^r), q_Y^!\,\mathrm{IB}(\Omega_Y^s)).$$

Proof. We have successively

$$\begin{aligned}
\mathrm{R}\mathcal{L}\,(q_X^{-1}\,\mathrm{IB}(\Omega_X^r), q_Y^!\,\mathrm{IB}(\Omega_Y^s)[d_Y]) &\simeq \mathrm{R}\mathcal{L}\,(q_X^{-1}\,\mathrm{IB}(\Omega_X^r), q_Y^!\,\mathrm{D}(\mathrm{IB}(\Omega_Y^{d_Y-s}))) &&(1)\\
&\simeq \mathrm{R}\mathcal{L}\,(q_X^{-1}\,\mathrm{IB}(\Omega_X^r), \mathrm{D}(q_Y^{-1}\,\mathrm{IB}(\Omega_Y^{d_Y-s})))\\
&\simeq \mathrm{R}\mathcal{L}\,(q_X^{-1}\,\mathrm{IB}(\Omega_X^r), \mathrm{R}\mathcal{L}\,(q_Y^{-1}\,\mathrm{IB}(\Omega_Y^{d_Y-s}), \omega_{X\times Y}))\\
&\simeq \mathrm{R}\mathcal{L}\,(\mathrm{IB}(\Omega_X^r)\hat{\boxtimes}^L\,\mathrm{IB}(\Omega_Y^{d_Y-s}), \omega_{X\times Y})\\
&\simeq \mathrm{R}\mathcal{L}\,(\mathrm{IB}(\Omega_{X\times Y}^{(r,d_Y-s)}), \omega_{X\times Y}) &&(2)\\
&\simeq \mathrm{D}(\mathrm{IB}(\Omega_{X\times Y}^{(r,d_Y-s)}))\\
&\simeq \mathrm{IB}(\Omega_{X\times Y}^{(d_X-r,s)})[d_{X\times Y}] &&(3)
\end{aligned}$$

where $\omega_{X\times Y}$ denotes the dualizing complex on $X \times Y$ for sheaves with values in $\mathcal{I}nd(\mathcal{B}an)$. Note that (1) and (3) follow from Theorem 5.3 and that (2) comes from Remark 6.2. $\square$

As a consequence, we may now give Proposition 5.7 and Proposition 5.6 their full generality.

Corollary 6.4. *Let X, Y be complex analytic manifolds of dimension d_X and d_Y. Assume K is a compact subset of X. Then,*

$$\mathrm{R}\Gamma_{K\times Y}(X \times Y; \mathrm{IB}(\Omega_{X\times Y}^{(d_X-r,s)})[d_X]) \simeq \mathrm{RL}\,(\mathrm{R}\Gamma(K; \mathrm{IB}(\Omega_X^r)); \mathrm{R}\Gamma(Y; \mathrm{IB}(\Omega_Y^s))).$$

Moreover, if X and Y are Stein manifolds and K is holomorphically convex in X, these complexes are concentrated in degree 0 and isomorphic to

$$\mathrm{IB}(\mathrm{L}_b(\Omega_X^r(K), \Omega_Y^s(Y))).$$

Proof. Transposing to sheaves with values in $\mathcal{I}nd(\mathcal{B}an)$ a classical result of the theory of abelian sheaves, we see that

$$\mathrm{R}\Gamma_{K \times Y}(X \times Y; \mathrm{R}\mathcal{L}\,(q_X^{-1}\mathcal{F}, q_Y^!\mathcal{G})) \simeq \mathrm{RL}\,(\mathrm{R}\Gamma(K; \mathcal{F}); \mathrm{R}\Gamma(Y; \mathcal{G}))$$

if $\mathcal{F}$ and $\mathcal{G}$ are objects of $\mathcal{S}hv(X; \mathcal{I}nd(\mathcal{B}an))$ and $\mathcal{S}hv(Y; \mathcal{I}nd(\mathcal{B}an))$. This formula combined with Theorem 6.3 gives the first part of the result. The second part follows from Proposition 3.6, Theorem 3.5 (using Remark 3.7, Theorem 2.13 and Proposition 1.13. $\qquad\square$

Corollary 6.5. *Let X, Y be complex analytic manifolds of dimension d_X and d_Y. Then,*

$$\mathrm{R}\Gamma(X \times Y; \mathrm{IB}(\Omega_{X \times Y}^{(d_X-r,s)})[d_X]) \simeq \mathrm{RL}\,(\mathrm{R}\Gamma_c(X; \mathrm{IB}(\Omega_X^r)), \mathrm{R}\Gamma(Y; \mathrm{IB}(\Omega_Y^s))).$$

Proof. This follows directly from the general isomorphism

$$\mathrm{R}\Gamma(X \times Y; \mathrm{R}\mathcal{L}\,(q_X^{-1}\mathcal{F}, q_Y^!\mathcal{G})) \simeq \mathrm{RL}\,(\mathrm{R}\Gamma_c(X; \mathcal{F}), \mathrm{R}\Gamma(Y; \mathcal{G}))$$

which holds for any objects $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{S}hv(X; \mathcal{I}nd(\mathcal{B}an))$ and $\mathcal{S}hv(Y; \mathcal{I}nd(\mathcal{B}an))$. $\qquad\square$

Lemma 6.6. *Let X be a complex analytic manifold of dimension d_X and let Y be a complex analytic submanifold of X of dimension d_Y. Then,*

$$LH^k(\mathrm{R}\Gamma_Y(\mathrm{IB}(\mathcal{O}_X))) \simeq 0$$

for $k \neq d_X - d_Y$.

Proof. Since the problem is local, it is sufficient to show that

$$LH^k(\mathrm{R}\Gamma_{\{0\} \times V}(U \times V; \mathrm{IB}(\mathcal{O}_{U \times V}))) \simeq 0$$

for $k \neq d_X - d_Y$ if U and V are Stein open neighborhoods of 0 in $\mathbb{C}^{d_X-d_Y}$ and $\mathbb{C}^{d_Y}$. In this situation, $\{0\}$ is a holomorphically convex compact subset of U and we get from Corollary 6.4 that

$$\mathrm{R}\Gamma_{\{0\} \times V}(U \times V; \mathrm{IB}(\mathcal{O}_{U \times V})[d_X - d_Y]) \simeq \mathrm{IB}(\mathrm{L}_b(\mathcal{O}_U(\{0\}), \mathcal{O}_V(V))).$$

The conclusion follows directly. $\qquad\square$

Theorem 6.7. *For any morphism of complex analytic manifolds $f : X \rightarrow Y$, we have a canonical isomorphism*

$$\mathrm{R}\mathcal{L}\,(f^{-1}\,\mathrm{IB}(\mathcal{O}_Y),\mathrm{IB}(\mathcal{O}_X)) \simeq \delta_f^{-1}\,\mathrm{R}\Gamma_{\Delta_f}\,\mathrm{IB}(\Omega_{X\times Y}^{(0,d_Y)})[d_Y]$$

where Δ_f is the graph of f in $X \times Y$ and $\delta_f : X \rightarrow X \times Y$ is the associated graph embedding. In particular,

$$LH^k(\mathrm{R}\mathcal{L}\,(f^{-1}\,\mathrm{IB}(\mathcal{O}_Y),\mathrm{IB}(\mathcal{O}_X))) = 0$$

for $k \neq 0$ and

$$R\mathcal{H}om\,(f^{-1}\,\mathrm{IB}(\mathcal{O}_Y),\mathrm{IB}(\mathcal{O}_X)) \simeq \mathcal{D}_{X\rightarrow Y}^\infty.$$

Proof. Using Theorem 6.3, we see that

$$\mathrm{IB}(\Omega_{X\times Y}^{(0,d_Y)})[d_Y] \simeq \mathrm{R}\mathcal{L}\,(q_Y^{-1}\,\mathrm{IB}(\mathcal{O}_Y),q_X^!\,\mathrm{IB}(\mathcal{O}_X)).$$

Applying $\delta_f^!$, we get successively

$$
\begin{aligned}
\delta_f^!\,\mathrm{IB}(\Omega_{X\times Y}^{(0,d_Y)})[d_Y] &\simeq \delta_f^!\mathrm{R}\mathcal{L}\,(q_Y^{-1}\,\mathrm{IB}(\mathcal{O}_Y),q_X^!\,\mathrm{IB}(\mathcal{O}_X))\\
&\simeq \mathrm{R}\mathcal{L}\,(\delta_f^{-1}q_Y^{-1}\,\mathrm{IB}(\mathcal{O}_Y),\delta_f^!q_X^!\,\mathrm{IB}(\mathcal{O}_X))\\
&\simeq \mathrm{R}\mathcal{L}\,((q_Y\circ\delta_f)^{-1}\,\mathrm{IB}(\mathcal{O}_Y),(q_X\circ\delta_f)^!\,\mathrm{IB}(\mathcal{O}_X))\\
&\simeq \mathrm{R}\mathcal{L}\,(f^{-1}\,\mathrm{IB}(\mathcal{O}_Y),\mathrm{IB}(\mathcal{O}_X)).
\end{aligned}
$$

This gives the first part of the result. To get the second one, it is sufficient to use Lemma 6.6, if we remember that, following [14], we have

$$\mathcal{D}_{X\rightarrow Y}^\infty \simeq \delta_f^{-1}\,\mathrm{R}\Gamma_{\Delta_f}\,\Omega_{X\times Y}^{(0,d_Y)}[d_Y].$$

$\square$

Corollary 6.8. *For any complex analytic manifold X of dimension d_X, we have a canonical isomorphism*

$$\mathrm{R}\mathcal{L}\,(\mathrm{IB}(\mathcal{O}_X),\mathrm{IB}(\mathcal{O}_X)) \simeq \delta^{-1}\,\mathrm{R}\Gamma_\Delta\,\mathrm{IB}(\Omega_{X\times X}^{(0,d_X)})[d_X]$$

where Δ is the diagonal of $X \times X$ and $\delta : X \rightarrow X \times X$ is the diagonal embedding. In particular,

$$LH^k(\mathrm{R}\mathcal{L}\,(\mathrm{IB}(\mathcal{O}_X),\mathrm{IB}(\mathcal{O}_X))) = 0$$

for $k \neq 0$ and

$$R\mathcal{H}om\,(\mathrm{IB}(\mathcal{O}_X),\mathrm{IB}(\mathcal{O}_X)) \simeq \mathcal{D}_X^\infty.$$

Remark 6.9. Note that the fact that continuous endomorphisms of $\mathcal{O}_X$ may be identified with partial differential operators of infinite order was conjectured by Sato and proved in [3]. The vanishing of the topological $\mathcal{E}xt^k$ ($k > 0$) is, to our knowledge, entirely new.

7   Reconstruction theorem

Let $\mathcal{R}$ be a ring on X with values in $\mathcal{I}nd(\mathcal{B}an)$ (i.e. a ring of the closed category $\mathcal{S}hv(X; \mathcal{I}nd(\mathcal{B}an))$ (see [16])). Denote by $\mathcal{M}od(\mathcal{R})$ the quasi-abelian category formed by $\mathcal{R}$-modules.

If $\mathcal{M}$, $\mathcal{N}$ are two $\mathcal{R}$-modules, one sees easily that $\mathcal{L}(\mathcal{M}, \mathcal{N})$ is endowed with both a structure of right $\mathcal{R}$-module and a compatible structure of left $\mathcal{R}$-module. These structures give two maps

$$\mathcal{L}(\mathcal{M}, \mathcal{N}) \xrightrightarrows{} \mathcal{L}(\mathcal{R}, \mathcal{L}(\mathcal{M}, \mathcal{N})).$$

As usual, we denote their equalizer by $\mathcal{L}_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$. In this way, we get a functor

$$\mathcal{L}_{\mathcal{R}}(\cdot, \cdot) : \mathcal{M}od(\mathcal{R})^{\mathrm{op}} \times \mathcal{M}od(\mathcal{R}) \to \mathcal{S}hv(X; \mathcal{I}nd(\mathcal{B}an))$$

which is clearly continuous on each variable and in particular left exact. Using the techniques of [16], one sees easily that $\mathcal{M}od(\mathcal{R})$ has enough injective objects and working as in [16, Proposition 2.3.10], one sees that the functor $\mathcal{L}_{\mathcal{R}}(\cdot, \cdot)$ has a right derived functor

$$\mathrm{R}\mathcal{L}_{\mathcal{R}}(\cdot, \cdot) : D^{-}(\mathcal{M}od(\mathcal{R}))^{\mathrm{op}} \times D^{+}(\mathcal{M}od(\mathcal{R})) \to D^{+}(\mathcal{S}hv(X; \mathcal{I}nd(\mathcal{B}an))).$$

Now, let $\mathcal{E}$ be a sheaf on X with values in $\mathcal{I}nd(\mathcal{B}an)$ and let $\mathcal{N}$ be an $\mathcal{R}$-module. Since $\mathcal{L}(\mathcal{E}, \mathcal{N})$ is canonically endowed with a structure of $\mathcal{R}$-module, we get a functor

$$\mathcal{L}(\cdot, \cdot) : \mathcal{S}hv(X; \mathcal{I}nd(\mathcal{B}an))^{\mathrm{op}} \times \mathcal{M}od(\mathcal{R}) \to \mathcal{M}od(\mathcal{R}).$$

One checks directly that this functor may be derived on the right by resolving the first argument by a complex of $K^{-}(\mathcal{S}hv(X; \mathcal{I}nd(\mathcal{B}an)))$ with components of the type

$$\bigoplus_{i \in I} (P_i)_{U_i}$$

(where P_i is a projective object of $\mathcal{I}nd(\mathcal{B}an)$ and U_i is an open subset of X) and the second argument by a complex of $K^{+}(\mathcal{M}od(\mathcal{R}))$ with flabby components. This gives us a derived functor

$$\mathrm{R}\mathcal{L}(\cdot, \cdot) : D^{-}(\mathcal{S}hv(X; \mathcal{I}nd(\mathcal{B}an)))^{\mathrm{op}} \times D^{+}(\mathcal{M}od(\mathcal{R})) \to D^{+}(\mathcal{M}od(\mathcal{R}))$$

which reduces to the usual $\mathrm{R}\mathcal{L}$ functor if we forget the $\mathcal{R}$-module structures.

Finally, recall that an object $\mathcal{M}$ of $D^{b}(\mathcal{M}od(\mathcal{R}))$ is perfect if there are integers $p \leq q$ such that for any $x \in X$ there is a neighborhood U of x with the property that $\mathcal{M}|_U$ is isomorphic to a complex of the type

$$0 \to \mathcal{R}|_U^{k_p} \to \cdots \to \mathcal{R}|_U^{k_q} \to 0$$

where the first component is in degree p and the last in degree q. We denote by $D_{pf}^b(\mathcal{M}od(\mathcal{R}))$ the triangulated subcategory of $D^b(\mathcal{M}od(\mathcal{R}))$ formed by perfect objects.

Proposition 7.1. *Let $\mathcal{N}$ be a sheaf on X with values in $\mathcal{I}nd(\mathcal{B}an)$ such that*

$$LH^k(\mathrm{R}\mathcal{L}\,(\mathcal{N},\mathcal{N})) = 0 \qquad (k \neq 0)$$

and let $\mathcal{R}$ be the ring $\mathcal{L}\,(\mathcal{N},\mathcal{N})$ of internal endomorphisms of $\mathcal{N}$. Then, $\mathcal{N}$ is an $\mathcal{R}$-module and the functor

$$\mathrm{R}\mathcal{L}_{\mathcal{R}}(\cdot,\mathcal{N}) : D_{pf}^b(\mathcal{M}od(\mathcal{R})) \longrightarrow D^b(\mathcal{S}hv(X;\mathcal{I}nd(\mathcal{B}an)))$$

is well-defined. Moreover, we have a canonical isomorphism

$$\mathrm{R}\mathcal{L}\,(\mathrm{R}\mathcal{L}_{\mathcal{R}}(\mathcal{M},\mathcal{N}),\mathcal{N}) \simeq \mathcal{M}$$

in $D(\mathcal{M}od(\mathcal{R}))$ for any $\mathcal{M} \in D_{pf}^b(\mathcal{M}od(\mathcal{R}))$. In particular, $\mathrm{R}\mathcal{L}_{\mathcal{R}}(\cdot,\mathcal{N})$ identifies $D_{pf}^b(\mathcal{M}od(\mathcal{R}))$ with a full triangulated subcategory of $D^b(\mathcal{S}hv(X;\mathcal{I}nd(\mathcal{B}an)))$.

Proof. For any $\mathcal{M} \in D_{pf}^b(\mathcal{M}od(\mathcal{R}))$, it is clear that

$$\mathrm{R}\mathcal{L}_{\mathcal{R}}(\mathcal{M},\mathcal{N}) \in D^b(\mathcal{S}hv(X;\mathcal{I}nd(\mathcal{B}an)))$$

since $\mathrm{R}\mathcal{L}_{\mathcal{R}}(\mathcal{R},\mathcal{N}) \simeq \mathcal{N}$. The canonical morphism

$$\mathcal{M} \,\hat{\otimes}^L \mathrm{R}\mathcal{L}_{\mathcal{R}}(\mathcal{M},\mathcal{N}) \rightarrow \mathcal{N}$$

induces by adjunction a morphism

$$\mathcal{M} \rightarrow \mathrm{R}\mathcal{L}\,(\mathrm{R}\mathcal{L}_{\mathcal{R}}(\mathcal{M},\mathcal{N}),\mathcal{N}).$$

If $\mathcal{M} \simeq \mathcal{R}$, $\mathrm{R}\mathcal{L}_{\mathcal{R}}(\mathcal{M},\mathcal{N}) \simeq \mathcal{N}$ and

$$\mathrm{R}\mathcal{L}\,(\mathrm{R}\mathcal{L}_{\mathcal{R}}(\mathcal{M},\mathcal{N}),\mathcal{N}) \simeq \mathrm{R}\mathcal{L}\,(\mathcal{N},\mathcal{N}) \simeq \mathcal{L}\,(\mathcal{N},\mathcal{N}) \simeq \mathcal{R}$$

and the preceding morphism is an isomorphism. It follows that it is also an isomorphism for $\mathcal{M} \simeq \mathcal{R}^k$ and, hence, if $\mathcal{M}$ is a complex of the type

$$0 \longrightarrow \mathcal{R}^{k_p} \longrightarrow \cdots \longrightarrow \mathcal{R}^{k_q} \longrightarrow 0.$$

Since any perfect complex is locally of this form, the conclusion follows easily. $\quad\square$

Let us consider the two functors

$$I_{\mathcal{V}} : \mathcal{V} \longrightarrow \mathcal{I}nd(\mathcal{B}an)$$
$$E \mapsto \quad \text{``}\varinjlim_{\substack{F \subset E \\ \dim F < +\infty}}\text{''} \quad F$$

and

$$L_{\mathcal{V}} : \mathcal{I}nd(\mathcal{B}an) \longrightarrow \mathcal{V}$$
$$\text{``}\varinjlim_{i \in \mathcal{I}}\text{''} \, E_i \mapsto \varinjlim_{i \in \mathcal{I}} E_i$$

where $\mathcal{V}$ denotes the category of $\mathbb{C}$-vector spaces. They are clearly linked by the adjunction formula

$$\text{Hom}\,(I_{\mathcal{V}}(E), F) \simeq \text{Hom}\,(E, L_{\mathcal{V}}(F))$$

and they are both exact. Moreover,

$$L_{\mathcal{V}} \circ I_{\mathcal{V}} = \text{id}\,.$$

For any sheaf E on X with values in $\mathcal{V}$, we denote $\tilde{I}_{\mathcal{V}}(E)$ the sheaf associated to the presheaf

$$U \mapsto I_{\mathcal{V}}(E(U)).$$

Similarly, to any sheaf F on X with values in $\mathcal{I}nd(\mathcal{B}an)$, we denote $\tilde{L}_{\mathcal{V}}(F)$ the sheaf

$$U \mapsto L_{\mathcal{V}}(F(U))$$

Working at the level of fibers, one checks easily that

$$\tilde{L}_{\mathcal{V}} \circ \tilde{I}_{\mathcal{V}} = \text{id}\,.$$

Proposition 7.2. *Let $\mathcal{N}$ be a sheaf on X with values in $\mathcal{I}nd(\mathcal{B}an)$ such that*

$$LH^k(R\mathcal{H}om\,(\mathcal{N}, \mathcal{N})) = 0 \qquad (k \neq 0)$$

and let $\mathcal{R}_{\mathcal{V}}$ be the ring $\mathcal{H}om\,(\mathcal{N}, \mathcal{N})$ of endomorphisms of $\mathcal{N}$. Then, $\mathcal{N}$ is an $\tilde{I}_{\mathcal{V}}(\mathcal{R}_{\mathcal{V}})$-module and the functor

$$\text{R}\mathcal{L}_{\tilde{I}_{\mathcal{V}}(\mathcal{R}_{\mathcal{V}})}(\tilde{I}_{\mathcal{V}}(\cdot), \mathcal{N}) : D_{pf}^b(\mathcal{M}od(\mathcal{R}_{\mathcal{V}})) \longrightarrow D^b(\mathcal{S}hv(X; \mathcal{I}nd(\mathcal{B}an)))$$

is well-defined. Moreover, we have a canonical isomorphism

$$R\mathcal{H}om\,(\text{R}\mathcal{L}_{\tilde{I}_{\mathcal{V}}(\mathcal{R}_{\mathcal{V}})}(\tilde{I}_{\mathcal{V}}(\mathcal{M}), \mathcal{N}), \mathcal{N}) \simeq \mathcal{M}$$

in $D(\mathcal{M}od(\mathcal{R}_{\mathcal{V}}))$ for any $\mathcal{M} \in D_{pf}^b(\mathcal{M}od(\mathcal{R}_{\mathcal{V}}))$.

In particular, $\text{R}\mathcal{L}_{\tilde{I}_{\mathcal{V}}(\mathcal{R}_{\mathcal{V}})}(\tilde{I}_{\mathcal{V}}(\cdot), \mathcal{N})$ identifies $D_{pf}^b(\mathcal{M}od(\mathcal{R}_{\mathcal{V}}))$ with a full triangulated subcategory of $D^b(\mathcal{S}hv(X; \mathcal{I}nd(\mathcal{B}an)))$.

Proof. Applying $\tilde{L}_\mathcal{V}$ to the morphism

$$\tilde{I}_\mathcal{V}(\mathcal{M}) \longrightarrow \mathrm{R}\mathcal{L}\left(\mathrm{R}\mathcal{L}_{\tilde{I}_\mathcal{V}(\mathcal{R}_\mathcal{V})}(\tilde{I}_\mathcal{V}(\mathcal{M}), \mathcal{N}), \mathcal{N}\right)$$

we get a canonical morphism

$$\mathcal{M} \longrightarrow R\mathcal{H}om\left(\mathrm{R}\mathcal{L}_{\tilde{I}_\mathcal{V}(\mathcal{R}_\mathcal{V})}(\tilde{I}_\mathcal{V}(\mathcal{M}), \mathcal{N}), \mathcal{N}\right)$$

since $\tilde{L}_\mathcal{V} \circ \mathrm{R}\mathcal{L} \simeq R\mathcal{H}om$. The conclusion follows by working as in the proof of Proposition 7.1. $\square$

Theorem 7.3. *Assume X is a complex analytic manifold of dimension d_X. Then, the sheaf $\mathrm{IB}(\mathcal{O}_X)$ is an $\tilde{I}_\mathcal{V}(\mathcal{D}_X^\infty)$-module and the functor*

$$\mathrm{R}\mathcal{L}_{\tilde{I}_\mathcal{V}(\mathcal{D}_X^\infty)}(\tilde{I}_\mathcal{V}(\cdot), \mathrm{IB}(\mathcal{O}_X)) : D_{pf}^b(\mathcal{M}od(\mathcal{D}_X^\infty)) \longrightarrow D^b(\mathcal{S}hv(X; \mathcal{I}nd(\mathcal{B}an)))$$

is well-defined. Moreover, we have a canonical isomorphism

$$R\mathcal{H}om\left(\mathrm{R}\mathcal{L}_{\tilde{I}_\mathcal{V}(\mathcal{D}_X^\infty)}(\tilde{I}_\mathcal{V}(\mathcal{M}), \mathrm{IB}(\mathcal{O}_X)), \mathrm{IB}(\mathcal{O}_X)\right) \simeq \mathcal{M}$$

in $D(\mathcal{M}od(\mathcal{D}_X^\infty))$ for any $\mathcal{M} \in D_{pf}^b(\mathcal{M}od(\mathcal{D}_X^\infty))$.

In particular, $\mathrm{R}\mathcal{L}_{\tilde{I}_\mathcal{V}(\mathcal{D}_X^\infty)}(\tilde{I}_\mathcal{V}(\cdot), \mathrm{IB}(\mathcal{O}_X))$ identifies $D_{pf}^b(\mathcal{M}od(\mathcal{D}_X^\infty))$ with a full triangulated subcategory of $D^b(\mathcal{S}hv(X; \mathcal{I}nd(\mathcal{B}an)))$.

Proof. Thanks to Corollary 6.8, this is an easy consequence of Proposition 7.2. $\square$

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