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by

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Abstract

This memoir is devided in three parts. In the first one, we introduce the notion of quasi-abelian category and link the homological algebra of these categories to that of their abelian envelopes. Note that quasi-abelian categories form a special class of non-abelian additive categories which contains in particular the category of locally convex topological vector spaces and the category of filtered abelian groups. In the second part, we define what we mean by an elementary quasi-abelian category and show that sheaves with values in such a category can be manipulated almost as easily as sheaves of abelian groups. In particular, we establish that the Poincaré-Verdier duality and the projection formula hold in this context. The third part is devoted to an application of the results obtained to the cases of filtered and topological sheaves.

Résumé

Ce mémoire est divisé en trois parties. Dans la première, nous introduisons la notion de catégorie quasi-abélienne et relions l'algèbre homologique de ces catégories à celle de leurs enveloppes abéliennes. Notons que les catégories quasi-abéliennes forment une classe spéciale de catégories additives non-abéliennes qui contient en particulier la catégorie des espaces vectoriels topologiques localement convexes et la catégorie des groupes abéliens filtrés. Dans la seconde partie, nous définissons ce que nous entendons par catégorie quasi-abélienne élémentaire et montrons que les faisceaux à valeurs dans une telle catégorie sont presque aussi aisés à manipuler que les faisceaux de groupes abéliens. En particulier, nous établissons que la dualité de Poincaré-Verdier et la formule de projection sont valides dans ce contexte. La troisième partie est consacrée à une application des résultats obtenus aux cas des faisceaux filtrés et topologiques.

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Introduction

To solve various problems of algebraic analysis, it would be very useful to have at hand a good cohomological theory of sheaves with values in categories like that of filtered modules or that of locally convex topological vector spaces. The problem to establish such a theory is twofold. A first difficulty comes form the fact that, since these categories are not abelian, the standard methods of homological algebra cannot be applied in the usual way. A second complication comes from the fact that we need to find conditions under which the corresponding cohomological theories of sheaves are well-behaved. This memoir grew out of the efforts of the author to understand how to modify the classical results in order to be able to treat such situations. The first two chapters deal separately with the two parts of the problem and the last one shows how to apply the theory developed to treat the cases of filtered and topological sheaves.

When we want to develop homological algebra for non abelian additive categories, a first approach is to show that the categories at hand may be endowed with structures of exact categories in the sense of D. Quillen [13]. Then, using Paragraph 1.3.22 of [1], it is possible to construct the corresponding derived category and to define what is right or left derived functor.

This approach was followed by G. Laumon in [9] to obtain interesting results for filtered $\mathcal{D}$-modules. We checked that it would also be possible to treat similarly the case of locally convex topological vector spaces. However, when one works out the details, it appears that a large part of the results does not come from the particular properties of filtered modules or locally convex topological vector spaces but instead come from the fact that the categories considered are exact categories of a very special kind. In fact, they are first examples of what we call quasi-abelian categories.

To provide a firm ground for applications to other situations, we have found it useful to devote Chapter 1 to a detailed study of the properties of these very special exact categories.

In Section 1.1, after a brief clarification of the notions of images, coimages and strict morphisms in additive categories, we give the axioms that such a category has to satisfy to be quasi-abelian. Next we show that a quasi-abelian category has a canonical exact structure. We conclude by giving precise definitions of the various exactness classes of additive functors between quasi-abelian categories. This is necessary since various exactness properties which are equivalent for abelian categories become distinct in the quasi-abelian case.

Section 1.2 is devoted to the construction of the derived category $\mathcal{D}(\mathcal{E})$ of a quasi-abelian category $\mathcal{E}$ and its two canonical t-structures. We introduce the two corresponding hearts $\mathcal{LH}(\mathcal{E})$ and $\mathcal{RH}(\mathcal{E})$ and we make a detailed study of the canonical embedding of $\mathcal{E}$ in $\mathcal{LH}(\mathcal{E})$. In particular, we show that the exact structure of $\mathcal{E}$ is induced by that of the abelian category $\mathcal{LH}(\mathcal{E})$ and that the derived category of $\mathcal{LH}(\mathcal{E})$ endowed with its canonical t-structure is equivalent to $\mathcal{D}(\mathcal{E})$ endowed with its left t-structure. Since the two canonical t-structures are exchanged by duality, it is not necessary to state explicitly the corresponding results for $\mathcal{RH}(\mathcal{E})$. Note that the canonical t-structures of $\mathcal{D}(\mathcal{E})$ and the abelian categories $\mathcal{LH}(\mathcal{E})$ and $\mathcal{RH}(\mathcal{E})$ cannot be defined for an arbitrary exact category and give first examples of the specifics of quasi-abelian categories. We end this section with a study of functors from a quasi-abelian category $\mathcal{E}$ to an abelian category $\mathcal{A}$ and show that $\mathcal{LH}(\mathcal{E})$ and $\mathcal{RH}(\mathcal{E})$ may in some sense be considered as abelian envelopes of $\mathcal{E}$.

In Section 1.3, we study how to derive an additive functor

$$F : \mathcal{E} \to \mathcal{F}$$

of quasi-abelian categories. After adapting the notions of F-projective and F-injective subcategories to our setting, we generalize the usual criterion for F to be left or right derivable. Next, we study various exactness properties of RF and relate them with the appropriate exactness properties of F. After having clarified how much of a functor is determined by its left or right derived functor and defined the relations of left and right equivalence for quasi-abelian functors, we show that, under mild assumptions, we can associate to F a functor

$$G : \mathcal{LH}(\mathcal{E}) \to \mathcal{LH}(\mathcal{F})$$

which has essentially the same left or right derived functor. Loosely speaking, the combination of this result and those of Section 1.2 shows that from the point of view of homological algebra we do not loose any information by replacing the quasi-abelian category $\mathcal{E}$ by the abelian category $\mathcal{LH}(\mathcal{E})$. We conclude this section by generalizing to quasi-abelian categories, the classical results on projective and injective objects. This leads us to make a careful distinction between projective (resp. injective) and strongly projective (resp. injective) objects of $\mathcal{E}$ and study how they are related with projective and injective objects of $\mathcal{LH}(\mathcal{E})$ or $\mathcal{RH}(\mathcal{E})$.

In Section 1.4, we deal with problems related to projective and inductive limits in quasi-abelian categories. First, we treat the case of products and show mainly that a quasi-abelian category $\mathcal{E}$ has exact (resp. strongly exact) products if and only if $\mathcal{LH}(\mathcal{E})$ (resp. $\mathcal{RH}(\mathcal{E})$) has exact products and the canonical functor

$$\mathcal{E} \to \mathcal{LH}(\mathcal{E}) \quad (\text{resp. } \mathcal{E} \to \mathcal{RH}(\mathcal{E}))$$

is product preserving. The case of coproducts is obtained by duality. After a detailed discussion of the properties of categories of projective or inductive systems of $\mathcal{E}$, we give conditions for projective or inductive limits to be computable in $\mathcal{E}$ as in $\mathcal{LH}(\mathcal{E})$. In this part, we have inspired ourselves from some methods of [3, 8]. We conclude by considering the special case corresponding to exact filtering inductive limits.

The last section of Chapter 1 is devoted to the special case of closed quasi-abelian categories (i.e. quasi-abelian categories with an internal tensor product, an internal homomorphism functor and a unit object satisfying appropriate axioms). We show mainly that in such a situation the category of modules over an internal ring is still quasi-abelian. Examples of such categories are numerous (e.g. filtered modules over a filtered ring, normed representations of a normed algebra) but the results obtained will also be useful to treat module over internal rings in a more abstract category like the category $\mathcal{W}$ defined in Chapter 3. We conclude by showing how a closed structure on $\mathcal{E}$ may induce, under suitable conditions, a closed structure on $\mathcal{LH}(\mathcal{E})$.

In Chapter 2 we study conditions on a quasi-abelian category $\mathcal{E}$ insuring that the category of sheaves with values in $\mathcal{E}$ is almost as easily to manipulate as the category of abelian sheaves.

In Section 2.1, we introduce the notions of quasi-elementary and elementary quasi-abelian categories and show that such categories are very easy to manipulate. First, we study the various natural notions of smallness in quasi-abelian categories. We also discuss strict generating sets which play for quasi-abelian categories the role played usually by the generating sets for abelian categories. This allows us to introduce the definitions of quasi-elementary and elementary categories and to show that if $\mathcal{E}$ is quasi-elementary then $\mathcal{LH}(\mathcal{E})$ is a category of functors with values in the category of abelian groups (this an analog of Freyd's result). We also show that the category of ind-objects of a small quasi-abelian category with enough projective objects is a basic example of an elementary quasi-abelian category. We conclude the section with a few results on closed elementary categories.

In Section 2.2, we show that that the category $\mathcal{S}hv(X;\mathcal{E})$ of sheaves on X with values in an elementary quasi-abelian category $\mathcal{E}$ is well-behaved. It is even endowed with internal operations if the category $\mathcal{E}$ is itself closed. Moreover, we show that

$$\mathcal{LH}(\mathcal{S}hv(X;\mathcal{E})) \approx \mathcal{S}hv(X;\mathcal{LH}(\mathcal{E}))$$

and thanks to the results in the preceding sections, we are reduced to work with sheaves in an elementary *abelian* category. Such sheaves where already studied in [14] where it is shown that they have most of the usual properties of abelian sheaves. In Section 2.3, we give further examples of how to extend to these sheaves results which are well known for abelian ones. In particular, we prove Poincaré-Verdier duality in this framework. We also prove that if $\mathcal{E}$ is closed and satisfies some mild assumptions then we can establish an internal projection formula and an internal Poincaré-Verdier duality formula by working almost as in the classical case.

Chapter 3 is devoted to applications to filtered and topological sheaves.

In Section 3.1, we study the category of filtered abelian groups and show that this is a closed elementary quasi-abelian with enough projective and injective objects. Its left abelian envelope $\mathcal{R}$ is identified with the category of graded modules over the graded ring $\mathbb{Z}[T]$ following an idea due to Rees. We also show that the category of separated filtered abelian sheaves is a quasi-elementary quasi-abelian category having $\mathcal{R}$ as its left abelian envelope. Since $\mathcal{R}$ is an elementary abelian category, the cohomological theory of sheaves developed in Chapter 2 may be applied to this category and gives a satisfying theory of filtered sheaves. Since most of the results in this section are easy consequences of the general theory, they are often given without proof.

Note that some of the results in this section where already obtained directly in specific situations by various authors (e.g. Illusie, Laumon, Rees, Saito, etc.). However, to our knownledge, the fact that all the classical cohomolgical formulas for abelian sheaves extend to filtered abelian sheaves was not yet fully established.

In Section 3.2, we show first that the category of semi-normed spaces in a closed quasi-abelian category with enough projective and injective objects which has the same left abelian envelope as the category of normed-spaces. Applying the results obtained before, we show that the category of ind-semi-normed spaces is a closed elementary quasi-abelian category and that its left abelian envelope $\mathcal{W}$ is a closed elementary abelian category. We also show that the category of locally convex topological vector spaces may be viewed as a (non full) subcategory of $\mathcal{W}$ and that through this identification, the categories of FN (resp. DFN) spaces appear as full subcategories of $\mathcal{W}$. Since the theory developed in Chapter 2 applies to $\mathcal{W}$, we feel that $\mathcal{W}$-sheaves provides a convenient notion of topological sheaves which is suitable for applications in algebraic analysis. Such applications are in preparation and will appear elsewhere.

Note that, in a private discussion some time ago, C. Houzel, conjectured that a category defined through the formula in Corollary 3.2.22 should be a good candidate to replace the category of locally convex topological vector spaces in problems dealing with sheaves and cohomology. He also suggested the name $\mathcal{W}$ since he expected this category to be related to the category of quotient bornological spaces introduced by Waelbroeck. We hope that the material in this paper will have convinced the reader that his insight was well-founded.

Before concluding this introduction, let us point out that discussions we had with M. Kashiwara on a first sketch of this paper lead him to a direct construction of the derived category of the category of FN (resp. DFN) spaces. These categories were used among other tools in [7] to prove very interesting formulas for quasi-equivariant $\mathcal{D}$-modules.

Note also that a study of the category of locally convex topological vector spaces along the lines presented here is being finalized by F. Prosmans. However, in this case the category is not elementary and one cannot treat sheaves with values in it along the lines of Chapter 2.

Throughout the paper, we assume the reader has a good knowledge of the theory of categories and of the homological algebra of abelian categories as exposed in standard reference works (e.g. [10, 11, 15] and [2, 4, 6, 16]). If someone would like an autonomous presentation of the basic facts concerning homological algebra of quasi-abelian categories, he may refer to [12] which was based on a preliminary version of Chapter 1.

Contents

Chapter 1

Quasi-Abelian Categories

1.1 Quasi-abelian categories and functors

Let $\mathcal{E}$ be an additive category with kernels and cokernels.

1.1.1 Images, coimages and strict morphisms

Definition 1.1.1. Let $f : E \longrightarrow F$ be a morphism of $\mathcal{E}$.

Following [4, 15], we define the *image* of f to be the kernel of the canonical map $F \longrightarrow \mathrm{Coker} f$. Dually, we define the *coimage* of f to be the cokernel of the canonical map $\mathrm{Ker} f \longrightarrow E$.

Obviously, f induces a canonical map

$$\mathrm{Coim} f \longrightarrow \mathrm{Im} f.$$

In general, this map is neither a monomorphism nor an epimorphism. When it is an isomorphism, we say that f is *strict*.

The following remark may help clarify the notion of strict morphism.

Remark 1.1.2.

(a) For any morphism $f : E \longrightarrow F$ of $\mathcal{E}$, the canonical morphism

$$\mathrm{Ker} f \longrightarrow E \qquad (\text{resp. } F \longrightarrow \mathrm{Coker} f)$$

is a strict monomorphism (resp. epimorphism).

(b) Let $f : E \longrightarrow F$ be a strict monomorphism (resp. epimorphism) of $\mathcal{E}$. Then f is a kernel (resp. cokernel) of

$$F \longrightarrow \mathrm{Coker} f \qquad (\text{resp. } \mathrm{Ker} f \longrightarrow E).$$

(c) A morphism f of $\mathcal{E}$ is strict if and only if

$$f = m \circ e$$

where m is a strict monomorphism and e is a strict epimorphism.

1.1.2 Definition of quasi-abelian categories

Definition 1.1.3. The category $\mathcal{E}$ is *quasi-abelian* if it satisfies the following dual axioms:

(QA) In a cartesian square

$$\begin{tikzcd} E \arrow[r, "f"] & F \\ E' \arrow[u] \arrow[r, "f'"'] & F' \arrow[u] \end{tikzcd}$$

where f is a strict epimorphism, f' is also a strict epimorphism.

(QA*) In a cocartesian square

$$\begin{tikzcd} E' \arrow[r, "f'"] & F' \\ E \arrow[u] \arrow[r, "f"'] & F \arrow[u] \end{tikzcd}$$

where f is a strict monomorphism, f' is also a strict monomorphism.

Until the end of this section, $\mathcal{E}$ will be assumed to be quasi-abelian.

Proposition 1.1.4. *Let $f : E \longrightarrow F$ be a morphism of $\mathcal{E}$. Then, in the canonical decomposition*

$$\begin{tikzcd} & \mathrm{Coim} f \arrow[dr, "k"] & \\ E \arrow[ur, "j"] \arrow[rr, "f"] & & F \end{tikzcd}$$

of f, j is a strict epimorphism and k is a monomorphism. Moreover, for any decomposition

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "h"'] & & F \\ & I^* \arrow[ur, "m"'] & \end{tikzcd}$$

of f where m is a monomorphism, there is a unique morphism

$$h' : \mathrm{Coim} f \longrightarrow I^*$$

making the diagram

$$\begin{tikzcd} & \mathrm{Coim} f \arrow[dr, "k"] \arrow[dd, "h'"] & \\ E \arrow[ur, "j"] \arrow[dr, "h"'] & & F \\ & I^* \arrow[ur, "m"'] & \end{tikzcd}$$

commutative.

Dually, in the canonical decomposition

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "k^*"'] & & F \\ & \mathrm{Im} f \arrow[ur, "j^*"'] & \end{tikzcd}$$

of f, k^* is an epimorphism and j^* is a strict monomorphism. Moreover, for any decomposition

$$\begin{tikzcd} & I \arrow[dr, "h"] & \\ E \arrow[ur, "e"] \arrow[rr, "f"] & & F \end{tikzcd}$$

of f where e is an epimorphism, there is a unique morphism

$$h' : I \longrightarrow \mathrm{Im} f$$

making the diagram

$$\begin{tikzcd} & I \arrow[dr, "h"] \arrow[dd, "h'"] & \\ E \arrow[ur, "e"] \arrow[dr, "k^*"'] & & F \\ & \mathrm{Im} f \arrow[ur, "j^*"'] & \end{tikzcd}$$

commutative.

Proof. Let

$$i : \mathrm{Ker} f \longrightarrow E$$

denote the canonical morphism. Since j is the cokernel of i, it is a strict epimorphism. Let us show that k is a monomorphism. Let $x : X \longrightarrow \mathrm{Coim} f$ be a morphism such that $k \circ x = 0$. Form the cartesian square

$$\begin{tikzcd} E \arrow[r, "j"] & \mathrm{Coim} f \\ X' \arrow[u, "x'"] \arrow[r, "j'"'] & X \arrow[u, "x"'] \end{tikzcd}$$

It follows from the fact that $\mathcal{E}$ is quasi-abelian that j' is a strict epimorphism. Since,

$$f \circ x' = k \circ j \circ x' = k \circ x \circ j' = 0,$$

there is a unique morphism $x'' : X' \longrightarrow \mathrm{Ker} f$ such that $i \circ x'' = x'$. From the relation

$$x \circ j' = j \circ x' = j \circ i \circ x'' = 0,$$

it follows that $x = 0$.

To prove the second part of the statement, note that, m being a monomorphism, it follows from the relation

$$m \circ h \circ i = f \circ i = 0,$$

that $h \circ i = 0$. Since j is the cokernel of i, there is a unique morphism

$$h' : \mathrm{Coim} f \longrightarrow I^*$$

such that

$$h' \circ j = h.$$

From the equality

$$k \circ j = m \circ h = m \circ h' \circ j,$$

it follows that

$$k = m \circ h'.$$

□

Corollary 1.1.5. *The canonical morphism*

$$\mathrm{Coim} f \longrightarrow \mathrm{Im} f$$

associated to a morphism $f : E \longrightarrow F$ of $\mathcal{E}$ is a bimorphism.

Remark 1.1.6. The preceding proposition shows in particular that the decomposition of f through $\mathrm{Coim} f$ (resp. $\mathrm{Im} f$) is in some sense the smallest (resp. greatest) decomposition of f as an epimorphism followed by a monomorphism. Hence, what we call $\mathrm{Im} f$ (resp. $\mathrm{Coim} f$) would be called $\mathrm{Coim} f$ (resp. $\mathrm{Im} f$) in [11]. Despite the good reasons for adopting Mitchell's point of view, we have chosen to stick to Grothendieck's definition which is more usual in the framework of additive categories.

1.1.3 Strict morphisms in quasi-abelian categories

Proposition 1.1.7. *The class of strict epimorphisms (resp. monomorphisms) of $\mathcal{E}$ is stable by composition.*

Proof. Let $u : E \longrightarrow F$ and $v : F \longrightarrow G$ be two strict epimorphisms and set $w = v \circ u$. We denote by $i_u : \mathrm{Ker} u \rightarrow E$ the canonical morphism and use similar notations for v and w. We get the commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Ker} w \arrow[r, "k"] \arrow[d, "i_w"] & \mathrm{Ker} v \arrow[d, "i_v"'] \\
\mathrm{Ker} u \arrow[ur, "j"] \arrow[r, "i_u"'] & E \arrow[r, "u"'] \arrow[dr, "w"'] & F \arrow[d, "v"'] \\
 & & G
\end{tikzcd}$$

One checks easily that the upper right square is cartesian. Since u is a strict epimorphism, it follows from the axioms that k is also a strict epimorphism. To conclude, it is sufficient to prove that w is a cokernel of i_w. Assume $f : E \longrightarrow X$ is a morphism such that $f \circ i_w = 0$. Since u is the cokernel of i_u and $f \circ i_u = 0$, there is a unique morphism $f' : F \longrightarrow X$ such that $f' \circ u = f$. Since k is an epimorphism, the equality

$$f' \circ i_v \circ k = f \circ i_w = 0$$

shows that $f' \circ i_v = 0$. Using the fact that v is the cokernel of i_v, we get a unique morphism $f'' : G \longrightarrow X$ such that $f'' \circ v = f'$. For this morphism, we get $f'' \circ w = f$ as requested. Moreover, w being an epimorphism, f'' is the only morphism satisfying this relation. □

Proposition 1.1.8. *Let*

$$\begin{tikzcd}
 & F \arrow[dr, "v"] & \\
E \arrow[ur, "u"] \arrow[rr, "w"'] & & G
\end{tikzcd}$$

be a commutative diagram in $\mathcal{E}$. Assume w is a strict epimorphism. Then, v is a strict epimorphism.

Dually, assume w is a strict monomorphism. Then, u is a strict monomorphism.

Proof. We will use the same commutative diagram as in the proof of the preceding proposition.

First, note that the square

$$\begin{tikzcd} E \oplus \mathrm{Ker} v \arrow[r, "(u\ i_v)"] \arrow[d, "(1\ 0)"'] & F \arrow[d, "v"] \\ E \arrow[r, "w"'] & G \end{tikzcd}$$

is cartesian. As a matter of fact, if the morphisms

$$X \xrightarrow{e} E, \quad X \xrightarrow{f} F$$

are such that

$$w \circ e = v \circ f,$$

then

$$v \circ (f - u \circ e) = 0$$

and there is $h : X \longrightarrow \mathrm{Ker} v$ such that

$$i_v \circ h = f - u \circ e.$$

It follows that for the morphism

$$\left(\begin{smallmatrix} e \\ h \end{smallmatrix}\right) : X \longrightarrow E \oplus \mathrm{Ker} v$$

we have

$$\left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} e \\ h \end{smallmatrix}\right) = e\ , \quad \left(\begin{smallmatrix} u & i_v \end{smallmatrix}\right) \left(\begin{smallmatrix} e \\ h \end{smallmatrix}\right) = f$$

and this is clearly the only morphism satisfying these conditions. It follows from the axioms and the fact that w is a strict epimorphism that

$$\left(\begin{smallmatrix} u & i_v \end{smallmatrix}\right) : E \oplus \mathrm{Ker} v \longrightarrow F$$

is a strict epimorphism.

Next, let $x : E \longrightarrow X$ be such that $x \circ i_v = 0$. It follows that

$$x \circ u \circ i_w = x \circ i_v \circ k = 0$$

and there is $x' : G \longrightarrow X$ such that

$$x \circ u = x' \circ w = x' \circ v \circ u.$$

Hence,

$$(x - x' \circ v) \circ u = 0$$

and since

$$(x - x' \circ v) \circ i_v = 0$$

we deduce from what precedes that $x = x' \circ v$. Since v is clearly an epimorphism, such an x' is unique. So, v is a cokernel of i_v and the conclusion follows. □

1.1.4 Strictly exact and coexact sequences

Definition 1.1.9. A null sequence

$$E' \xrightarrow{e'} E \xrightarrow{e''} E''$$

of $\mathcal{E}$ is *strictly exact (resp. coexact)* if e' (resp. e'') is strict and if the canonical morphism

$$\operatorname{Im} e' \longrightarrow \operatorname{Ker} e''$$

is an isomorphism. More generally, a sequence

$$E_1 \xrightarrow{e^1} \cdots \xrightarrow{e^{n-1}} E_n \quad (n \geq 3)$$

is strictly exact (resp. coexact) if each of the subsequences

$$E_i \xrightarrow{e^i} E_{i+1} \xrightarrow{e^{i+1}} E_{i+2} \quad (1 \leq i \leq n-2)$$

is strictly exact (resp. coexact).

Remark 1.1.10. It follows from the preceding definition that strict exactness and strict coexactness are dual notions which are in general not equivalent. However, a short sequence

$$0 \longrightarrow E \xrightarrow{u} F \xrightarrow{v} G \longrightarrow 0$$

is strictly exact if and only if u is a kernel of v and v is a cokernel of u. Hence, such a sequence is strictly exact if and only if it is strictly coexact.

Remark 1.1.11. Thanks to the results in the preceding subsections, it is easily seen that the category $\mathcal{E}$ endowed with the class of short strictly exact sequences forms an exact category in the sense of [13].

1.1.5 Exactness classes of quasi-abelian functors

Since there are two kinds of exact sequences in a quasi-abelian category, there are more exactness classes of functors than in the abelian case. All these various classes are needed in the rest of the paper. Hence, we will define them carefully in this section.

Let us first consider left exactness.

Definition 1.1.12. Let

$$F : \mathcal{E} \longrightarrow \mathcal{F}$$

be an additive functor.

We say that F is *left exact* if it transforms any strictly exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

of $\mathcal{E}$ into the strictly exact sequence

$$0 \longrightarrow F(E') \longrightarrow F(E) \longrightarrow F(E'')$$

of $\mathcal{F}$. Equivalently, F is left exact if it preserves kernels of strict morphisms.

We say that F is *strongly left exact* if it transforms any strictly exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E''$$

of $\mathcal{E}$ into the strictly exact sequence

$$0 \longrightarrow F(E') \longrightarrow F(E) \longrightarrow F(E'')$$

of $\mathcal{F}$. Equivalently, F is strongly left exact if it preserves kernels of arbitrary morphisms.

Finally, we say that F is *regular* if it transforms a strict morphism into a strict morphism and *regularizing* if it transforms an arbitrary morphism into a strict morphism.

Remark 1.1.13. Some authors have defined a left exact functor between arbitrary finitely complete categories to be a functor which preserves all finite projective limits. This definition coincides obviously with our notion of strongly left exact functor.

Other definitions of left exactness can also be introduced. They are clarified in Proposition 1.1.15 the proof of which is left to the reader.

Definition 1.1.14. Let

$$F : \mathcal{E} \longrightarrow \mathcal{F}$$

be an additive functor. Let S denote a null sequence of the form

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E''$$

and let $F(S)$ denote the null sequence

$$0 \longrightarrow F(E') \longrightarrow F(E) \longrightarrow F(E'').$$

We shall distinguish four notions of left exactness for the functor F. They are defined in the following table by the exactness property of $F(S)$ which follows from a given exactness property of S.

F	S	$F(S)$
LL left exact	strictly exact	strictly exact
LR left exact	strictly exact	strictly coexact
RL left exact	strictly coexact	strictly exact
RR left exact	strictly coexact	strictly coexact

Proposition 1.1.15. *Let*

$$F : \mathcal{E} \to \mathcal{F}$$

be an additive functor between quasi-abelian categories.

(LL) The functor F is LL left exact if and only if it is strongly left exact.

(LR) The functor F is LR left exact if and only if it is strongly left exact and regularizing.

(RL) The functor F is RL left exact if and only if it is left exact.

(RR) The functor F is RR left exact if and only if it is left exact and regular.

Remark 1.1.16. Note also that with the notation of the preceding proposition, F is LL left exact if and only if it is RL left exact and transforms a monomorphism into a monomorphism. Similarly, F is LR left exact if and only if it is RR left exact and transforms a monomorphism into a strict monomorphism.

Having clarified the various notions of left exactness, we can treat right exactness by duality.

Definition 1.1.17. Let

$$F : \mathcal{E} \to \mathcal{F}$$

be an additive functor.

We say that F is *right exact* if it transforms any strictly (co)exact sequence

$$0 \to E' \to E \to E'' \to 0$$

of $\mathcal{E}$ into the strictly coexact sequence

$$F(E') \to F(E) \to F(E'') \to 0$$

of $\mathcal{F}$. Equivalently, F is right exact if it preserves cokernels of strict morphisms.

We say that F is *strongly right exact* if it transforms any strictly coexact sequence

$$E' \to E \to E'' \to 0$$

of $\mathcal{E}$ into the strictly coexact sequence

$$F(E') \to F(E) \to F(E'') \to 0$$

of $\mathcal{F}$. Equivalently, F is strongly right exact if it preserves cokernels of arbitrary morphisms.

Finally let us introduce the various classes of exact functors.

Definition 1.1.18. Let

$$F : \mathcal{E} \to \mathcal{F}$$

be an additive functor.

The functor F is *exact* if it transforms any strictly (co)exact sequence

$$0 \to E' \to E \to E'' \to 0$$

of $\mathcal{E}$ into the strictly (co)exact sequence

$$0 \to F(E') \to F(E) \to F(E'') \to 0$$

of $\mathcal{F}$. Equivalently, F is exact if it is both left exact and right exact.

The functor F is *strongly exact* if it transforms any strictly exact (resp. coexact) sequence

$$E' \to E \to E''$$

of $\mathcal{E}$ into the strictly exact (resp. coexact) sequence

$$F(E') \to F(E) \to F(E'')$$

of $\mathcal{F}$. Equivalently, F is strongly exact if it is both strongly left exact and strongly right exact.

The functor F is *strictly exact* (resp. *strictly coexact*) if it transforms any strictly exact (resp. coexact) sequence

$$E' \to E \to E''$$

of $\mathcal{E}$ into a strictly exact (resp. coexact) sequence

$$F(E') \to F(E) \to F(E'')$$

of $\mathcal{F}$.

1.2 Derivation of quasi-abelian categories

1.2.1 The category $\mathcal{K}(\mathcal{E})$ and its canonical t-structures

In this subsection, we assume that $\mathcal{E}$ is an additive category with kernels and cokernels. Since $\mathcal{E}$ is additive, it is well-known that the associated category $\mathcal{K}(\mathcal{E})$ of complexes modulo homotopy is a triangulated category. Here we will show that it is also canonically endowed with two t-structures which are exchanged by duality.

Definition 1.2.1. A null sequence

$$E' \longrightarrow E \longrightarrow E''$$

of $\mathcal{E}$ is *split* if, for any object X of $\mathcal{E}$, the associated sequence

$$\operatorname{Hom}_{\mathcal{E}}(X, E') \longrightarrow \operatorname{Hom}_{\mathcal{E}}(X, E) \longrightarrow \operatorname{Hom}_{\mathcal{E}}(X, E'')$$

is an exact sequence of abelian groups. Dually, it is *cosplit* if, for any object X of $\mathcal{E}$, the associated sequence

$$\operatorname{Hom}_{\mathcal{E}}(E'', X) \longrightarrow \operatorname{Hom}_{\mathcal{E}}(E, X) \longrightarrow \operatorname{Hom}_{\mathcal{E}}(E', X)$$

is an exact sequence of abelian groups.

A complex E of $\mathcal{E}$ is *split (resp. cosplit) in degree* n if the sequence

$$E^{n-1} \xrightarrow{d^{n-1}} E^n \xrightarrow{d^n} E^{n+1}$$

is split (resp. cosplit).

A complex is *split (resp. cosplit)* if it is split (resp. cosplit) in each degree.

Remark 1.2.2.

(a) A null sequence

$$E' \xrightarrow{e'} E \xrightarrow{e''} E''$$

of $\mathcal{E}$ is split if and only if the associated short sequence

$$0 \longrightarrow \operatorname{Ker} e' \longrightarrow E \longrightarrow \operatorname{Ker} e'' \longrightarrow 0$$

splits. In particular, a sequence may be split without being cosplit.

(b) A complex E of $\mathcal{E}$ is split if and only if it is homotopically equivalent to 0. Hence, E is split if and only if it is cosplit.

Lemma 1.2.3. *Any object E of $\mathcal{K}(\mathcal{E})$ may be embedded in a distinguished triangle*

$$E^{\leq 0} \longrightarrow E \longrightarrow E^{>0} \xrightarrow{+1}$$

of $\mathcal{K}(\mathcal{E})$ where $E^{\leq 0}$ is the complex

$$\cdots E^{-2} \longrightarrow E^{-1} \longrightarrow \mathrm{Ker} d_E^0 \longrightarrow 0$$

(with $\mathrm{Ker} d_E^0$ in degree 0) and $E^{>0}$ is the complex

$$0 \longrightarrow \mathrm{Ker} d_E^0 \longrightarrow E^0 \longrightarrow E^1 \cdots$$

(with E^0 in degree 0).

Proof. Denote by i the canonical morphism from $\mathrm{Ker} d_E^0$ to E^0 and let $u : E^{\leq 0} \longrightarrow E$ be the morphism defined by

$$u^n = \begin{cases} 1 & \text{for} \quad n < 0, \\ i & \text{for} \quad n = 0, \\ 0 & \text{for} \quad n > 0. \end{cases}$$

By definition, the mapping cone M of u is the complex:

$$\cdots E^{-1} \oplus E^{-2} \xrightarrow{\left(\begin{smallmatrix} -d & 0 \\ 1 & d \end{smallmatrix}\right)} \mathrm{Ker} d_E^0 \oplus E^{-1} \xrightarrow{(i \; d)} E^0 \xrightarrow{d} E^1 \cdots$$

(with E^0 in degree 0). Let $\alpha : E^{>0} \longrightarrow M$ and $\beta : M \longrightarrow E^{>0}$ be the morphisms defined respectively by:

$$\alpha^n = \begin{cases} 0 & \text{for} \quad n < -1, \\ \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) & \text{for} \quad n = -1, \\ 1 & \text{for} \quad n > -1, \end{cases} \qquad \text{and} \qquad \beta^n = \begin{cases} 0 & \text{for} \quad n < -1, \\ \left(\begin{smallmatrix} 1 & d \end{smallmatrix}\right) & \text{for} \quad n = -1, \\ 1 & \text{for} \quad n > -1. \end{cases}$$

One checks easily that

$$\beta \circ \alpha = \mathrm{id}_{E^{>0}} \quad \text{and} \quad \alpha \circ \beta - \mathrm{id}_M = d_M \circ h + h \circ d_M$$

where h is the homotopy defined by:

$$h^n = \begin{cases} \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) & \text{for} \quad n < 0, \\ 0 & \text{for} \quad n \geq 0. \end{cases}$$

Therefore, $E^{>0}$ is homotopically equivalent to M and the conclusion follows. $\square$

Proposition 1.2.4. *Let $\mathcal{K}^{\leq 0}(\mathcal{E})$ (resp. $\mathcal{K}^{\geq 0}(\mathcal{E})$) denote the full subcategory of $\mathcal{K}(\mathcal{E})$ formed by the complexes which are split in each strictly positive (resp. strictly negative) degree. Then, the pair*

$$(\mathcal{K}^{\leq 0}(\mathcal{E}), \mathcal{K}^{\geq 0}(\mathcal{E}))$$

defines a t-structure on $\mathcal{K}(\mathcal{E})$.

Proof. Thanks to the preceding lemma, we need only to prove that

$$\operatorname{Hom}_{\mathcal{K}(\mathcal{E})}(E, F) = 0$$

for $E \in \mathcal{K}^{\leq 0}(\mathcal{E})$ and $F \in \mathcal{K}^{>0}(\mathcal{E})$. Using the preceding lemma, we get the two distinguished triangles

$$E^{\leq 0} \longrightarrow E \longrightarrow E^{>0} \xrightarrow{+1} \qquad F^{\leq 0} \longrightarrow F \longrightarrow F^{>0} \xrightarrow{+1} .$$

Thanks to Remark 1.2.2, our assumptions show that $E^{>0} \simeq 0$ and that $F^{\leq 0} \simeq 0$ in $\mathcal{K}(\mathcal{E})$ and the preceding distinguished triangles allow us to conclude that $E^{\leq 0} \simeq E$ and $F \simeq F^{>0}$. Since one checks easily that

$$\operatorname{Hom}_{\mathcal{K}(\mathcal{E})}(E^{\leq 0}, F^{>0}) = 0,$$

the proof is complete. □

Definition 1.2.5. We call the canonical t-structure studied in the preceding proposition the *left t-structure of* $\mathcal{K}(\mathcal{E})$. We denote by $\mathcal{LK}(\mathcal{E})$ its heart and by LK^n the corresponding cohomology functors.

Proposition 1.2.6. *The truncation functors $\tau^{\leq n}$, $\tau^{\geq n}$ for the left t-structure of $\mathcal{K}(\mathcal{E})$ associate respectively to a complex E the complex*

$$\dots E^{n-2} \longrightarrow E^{n-1} \longrightarrow \operatorname{Ker} d_E^n \longrightarrow 0$$

(with $\operatorname{Ker} d_E^n$ in degree n) and the complex

$$0 \longrightarrow \operatorname{Ker} d_E^{n-1} \longrightarrow E^{n-1} \longrightarrow E^n \dots$$

(with E^n in degree n). Hence, $LK^n(E)$ is the complex

$$0 \longrightarrow \operatorname{Ker} d_E^{n-1} \longrightarrow E^{n-1} \longrightarrow \operatorname{Ker} d_E^n \longrightarrow 0$$

where $\operatorname{Ker} d_E^n$ is in degree 0.

Corollary 1.2.7. *The category $\mathcal{LK}(\mathcal{E})$ is equivalent to the full subcategory of $\mathcal{K}(\mathcal{E})$ consisting of complexes of the form*

$$0 \longrightarrow \mathrm{Ker} f \longrightarrow E \xrightarrow{f} F \longrightarrow 0$$

(F in degree 0).

Definition 1.2.8. An object A of $\mathcal{LK}(\mathcal{E})$ *is represented by the morphism*

$$f : E \longrightarrow F$$

if it is isomorphic to the complex

$$0 \longrightarrow \mathrm{Ker} f \longrightarrow E \xrightarrow{f} F \longrightarrow 0$$

where F is in degree 0.

Remark 1.2.9. Through the canonical equivalence

$$\mathcal{K}(\mathcal{E})^{\mathrm{op}} \approx \mathcal{K}(\mathcal{E}^{\mathrm{op}}),$$

the left t-structure of $\mathcal{K}(\mathcal{E}^{\mathrm{op}})$ gives a second t-structure on $\mathcal{K}(\mathcal{E})$. We call it the *right t-structure of* $\mathcal{K}(\mathcal{E})$. We denote by $\mathcal{RK}(\mathcal{E})$ its heart and by RK^n the corresponding cohomology functors. The reader will easily dualize the preceding discussion and make the link between the right t-structure of $\mathcal{K}(\mathcal{E})$ and cosplit sequences of $\mathcal{E}$.

1.2.2 The category $\mathcal{D}(\mathcal{E})$ and its canonical t-structures

In this subsection, we assume that the category $\mathcal{E}$ is quasi-abelian.

Definition 1.2.10. A complex E of $\mathcal{E}$ is *strictly exact (resp. coexact) in degree* n if the sequence

$$E^{n-1} \xrightarrow{d^{n-1}} E^n \xrightarrow{d^n} E^{n+1}$$

is strictly exact (resp. coexact).

A complex of $\mathcal{E}$ is *strictly exact (resp. coexact)* if it is strictly exact (resp. coexact) in each degree.

Remark 1.2.11. A complex of $\mathcal{E}$ is strictly exact if and only it is strictly coexact.

Lemma 1.2.12. *Two isomorphic objects of $\mathcal{K}(\mathcal{E})$ are simultaneously strictly exact in degree n.*

Proof. Let E, F be two isomorphic objects of $\mathcal{K}(\mathcal{E})$ and assume E is strictly exact in degree n. Applying LK^n, we see that the complexes

$$0 \longrightarrow \operatorname{Ker} d_E^{n-1} \xrightarrow{i_E^{n-1}} E^{n-1} \xrightarrow{\delta_E^{n-1}} \operatorname{Ker} d_E^n \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Ker} d_F^{n-1} \xrightarrow{i_F^{n-1}} F^{n-1} \xrightarrow{\delta_F^{n-1}} \operatorname{Ker} d_F^n \longrightarrow 0$$

are homotopically equivalent. Let

$$\alpha : LK^n(E) \longrightarrow LK^n(F) \quad \text{and} \quad \beta : LK^n(F) \longrightarrow LK^n(E)$$

be two morphisms of complexes such that

$$\mathrm{id}_{LK^n(F)} - \alpha \circ \beta = h \circ d_{LK^n(F)} + d_{LK^n(F)} \circ h$$

where h is a homotopy. We know that $LK^n(E)$ is strictly exact and we have to show that so is $LK^n(F)$. To this end, it is sufficient to show that δ_F^{n-1} is a cokernel of i_F^{n-1}.

It is clear that δ_F^{n-1} is an epimorphism. As a matter of fact, if $g : \operatorname{Ker} d_F^n \longrightarrow X$ satisfies $g \circ \delta_F^{n-1} = 0$, it follows from the relation

$$\mathrm{id}_{\operatorname{Ker} d_F^n} - \alpha^0 \circ \beta^0 = \delta_F^{n-1} \circ h^0$$

that $g = g \circ \alpha^0 \circ \beta^0$. Since

$$g \circ \alpha^0 \circ \delta_E^{n-1} = g \circ \delta_F^{n-1} \circ \alpha^{-1} = 0$$

and δ_E^{n-1} is an epimorphism, $g \circ \alpha^0 = 0$ and the conclusion follows.

Assume now $f : F^{n-1} \longrightarrow A$ is a morphism such that $f \circ i_F^{n-1} = 0$. From this equality, we deduce that $f \circ \alpha^{-1} \circ i_E^{n-1} = 0$. Since δ_E^{n-1} is a cokernel of i_E^{n-1}, we get a unique morphism $f' : \operatorname{Ker} d_E^n \longrightarrow X$ such that $f' \circ \delta_E^{n-1} = f \circ \alpha^{-1}$. Therefore,

$$f' \circ \beta^0 \circ \delta_F^{n-1} = f' \circ \delta_E^{n-1} \circ \beta^{-1} = f \circ \alpha^{-1} \circ \beta^{-1}.$$

Since

$$\mathrm{id}_{F^{n-1}} - \alpha^{-1} \circ \beta^{-1} = h^0 \circ \delta_F^{n-1} + i_F^{n-1} \circ h^{-1},$$

we get

$$f - f' \circ \beta^0 \circ \delta_F^{n-1} = f \circ h^0 \circ \delta_F^{n-1} + f \circ i_F^{n-1} \circ h^{-1}.$$

Since $f \circ i_F^{n-1} = 0$, we finally get

$$f = (f' \circ \beta^0 + f \circ h^0) \circ \delta_F^{n-1}$$

and this concludes the proof. □

Proposition 1.2.13. *Let*

$$E \xrightarrow{u} F \xrightarrow{v} G \xrightarrow{w} E[1]$$

be a distinguished triangle of $\mathcal{K}(\mathcal{E})$. Assume E and G are strictly exact in degree n. Then F is also strictly exact in the same degree.

Proof. Thanks to the preceding lemma, we may assume F is the mapping cone of

$$-w[-1] : G[-1] \longrightarrow E.$$

Hence, $F^k = E^k \oplus G^k$ and

$$d_F^k = \begin{pmatrix} d_E^n & -w^n \\ 0 & d_G^n \end{pmatrix}.$$

Denote by $\delta : \mathrm{Ker} d_E^n \oplus G^{n-1} \longrightarrow \mathrm{Ker} d_F^n$ the morphism induced by

$$\begin{pmatrix} 1 & -w^{n-1} \\ 0 & d_G^{n-1} \end{pmatrix} : E^n \oplus G^{n-1} \longrightarrow E^n \oplus G^n$$

One checks easily that the square

$$\begin{tikzcd} G^{n-1} \arrow[r, "\delta_G^{n-1}"] & \mathrm{Ker} d_G^n \\ \mathrm{Ker} d_E^n \oplus G^{n-1} \arrow[u, "q_{G^{n-1}}"] \arrow[r, "\delta"'] & \mathrm{Ker} d_F^n \arrow[u, "q_{G^n}"'] \end{tikzcd}$$

is cartesian. Hence, it follows from our assumptions that δ is a strict epimorphism. Since

$$d_E^{n-1} : E^{n-1} \longrightarrow \mathrm{Ker} d_E^n$$

is a strict epimorphism, so is

$$d_E^{n-1} \oplus \mathrm{id}_{G^{n-1}} : E^{n-1} \oplus G^{n-1} \longrightarrow \mathrm{Ker} d_E^n \oplus G^{n-1}.$$

By composing with δ, we see, by Lemma 1.1.7, that

$$d_F^{n-1} : F^{n-1} \longrightarrow \mathrm{Ker} d_F^n$$

is a strict epimorphism and the proof is complete. □

Corollary 1.2.14. *Strictly exact complexes form a saturated null system in $\mathcal{K}(\mathcal{E})$.*

Proof. The axioms for a null system are easily checked thanks to the preceding proposition. Since it is clear that a direct sum of two complexes is strictly exact if and only if each summand is itself strictly exact, the saturation is also clear. □

Definition 1.2.15. We denote by $\mathcal{N}(\mathcal{E})$ the full subcategory of $\mathcal{K}(\mathcal{E})$ formed by the complexes which are strictly exact. Since $\mathcal{N}(\mathcal{E})$ is a null system, we may define the *derived category of* $\mathcal{E}$ by the formula:

$$\mathcal{D}(\mathcal{E}) = \mathcal{K}(\mathcal{E})/\mathcal{N}(\mathcal{E}).$$

A morphism of $\mathcal{K}(\mathcal{E})$ which has a strictly exact mapping cone is called a *strict quasi-isomorphism.*

Lemma 1.2.16. *Let $\mathcal{T}$ be a triangulated category endowed with a t-structure*

$$(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}).$$

Assume $\mathcal{N}$ is a saturated null system of $\mathcal{T}$. Denote by

$$Q : \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{N}$$

the canonical functor and by $(\mathcal{T}/\mathcal{N})^{\leq 0}$ (resp. $(\mathcal{T}/\mathcal{N})^{\geq 0}$) the essential image of $Q_{|\mathcal{T}^{\leq 0}}$ (resp. $Q_{|\mathcal{T}^{\geq 0}}$). Then,

$$((\mathcal{T}/\mathcal{N})^{\leq 0}, (\mathcal{T}/\mathcal{N})^{\geq 0})$$

is a t-structure on $\mathcal{T}/\mathcal{N}$ if and only if for any distinguished triangle

$$X_1 \longrightarrow X_0 \longrightarrow N \xrightarrow{+1}$$

where $X_1 \in \mathcal{T}^{\geq 1}$, $X_0 \in \mathcal{T}^{\leq 0}$ and $N \in \mathcal{N}$, we have X_1, $X_0 \in \mathcal{N}$.

Proof. Let us proof that the condition is necessary. Consider a triangle

$$X_1 \longrightarrow X_0 \longrightarrow N \xrightarrow{+1}$$

of $\mathcal{T}$ where $X_1 \in \mathcal{T}^{\geq 1}$, $X_0 \in \mathcal{T}^{\leq 0}$ and $N \in \mathcal{N}$. It gives rise to the triangle

$$Q(X_1) \longrightarrow Q(X_0) \longrightarrow Q(N) \xrightarrow{+1}$$

of $\mathcal{T}/\mathcal{N}$. Since $Q(N) \simeq 0$,

$$Q(X_1) \longrightarrow Q(X_0)$$

is an isomorphism in $\mathcal{T}/\mathcal{N}$. Its inverse belongs to

$$\mathrm{Hom}_{\mathcal{T}/\mathcal{N}}(Q(X_0), Q(X_1))$$

and our assumption shows that it is the zero morphism. Therefore, both $Q(X_0)$ and $Q(X_1)$ are isomorphic to 0 in $\mathcal{T}/\mathcal{N}$ and the conclusion follows from the fact that $\mathcal{N}$ is a saturated null system.

To prove that the condition is sufficient, we have only to show that

$$\mathrm{Hom}_{\mathcal{T}/\mathcal{N}}(Q(X_0), Q(X_1)) = 0$$

for $X_0 \in \mathcal{T}^{\leq 0}$, $X_1 \in \mathcal{T}^{\geq 1}$. A morphism from $Q(X_0)$ to $Q(X_1)$ is represented by a diagram

$$\begin{tikzcd} & Y \arrow[dl, "s"'] \arrow[dr, "a"] & \\ X_0 & & X_1 \end{tikzcd}$$

where s, a are morphisms in $\mathcal{T}$, s being an $\mathcal{N}$-quasi-isomorphism. Thus, in $\mathcal{T}$, we have a distinguished triangle

$$Y \xrightarrow{s} X_0 \longrightarrow N \xrightarrow{+1}$$

where $N \in \mathcal{N}$. By the properties of t-structures, we also have in $\mathcal{T}$ a distinguished triangle

$$Y_0 \xrightarrow{t} Y \longrightarrow Y_1 \xrightarrow{+1}$$

where $Y_0 \in \mathcal{T}^{\leq 0}$ and $Y_1 \in \mathcal{T}^{\geq 1}$. Let us embed $s \circ t$ in a distinguished triangle

$$Y_0 \xrightarrow{s \circ t} X_0 \longrightarrow N_0 \xrightarrow{+1} .$$

Applying the axiom of the octahedron, we get a distinguished triangle

$$Y_1 \longrightarrow N_0 \longrightarrow N \xrightarrow{+1} .$$

By our assumptions, both Y_1 and N_0 are objects of $\mathcal{N}$. Therefore, t is an $\mathcal{N}$-quasi-isomorphism. Hence, we get the commutative diagram:

$$\begin{tikzcd} & Y_0 \arrow[d, "t"] \arrow[ddl, bend right, "s \circ t"'] \arrow[ddr, bend left, "0"] & \\ & Y \arrow[dl, "s"'] \arrow[dr, "a"] & \\ X_0 & & X_1 \end{tikzcd}$$

where the map from Y_0 to X_1 is 0 since

$$\mathrm{Hom}_{\mathcal{T}}(Y_0, X_1) = 0.$$

The conclusion follows easily. $\square$

Definition 1.2.17. Thanks to Proposition 1.2.13, the preceding lemma shows that the left t-structure on $\mathcal{K}(\mathcal{E})$ induces a canonical t-structure on $\mathcal{D}(\mathcal{E})$, we call it the *left t-structure of* $\mathcal{D}(\mathcal{E})$. We denote $\mathcal{LH}(\mathcal{E})$ its heart and

$$LH^n : \mathcal{D}(\mathcal{E}) \longrightarrow \mathcal{LH}(\mathcal{E})$$

the corresponding cohomology functors.

Proposition 1.2.18. *The truncation functors $\tau^{\leq n}$, $\tau^{\geq n}$ associated to the left t-structure of $\mathcal{D}(\mathcal{E})$ send a complex E respectively to*

$$\cdots \longrightarrow E^{n-2} \longrightarrow E^{n-1} \longrightarrow \mathrm{Ker} d_E^n \longrightarrow 0$$

($\mathrm{Ker} d_E^n$ *in degree n) and to*

$$0 \longrightarrow \mathrm{Coim} d_E^{n-1} \longrightarrow E^n \longrightarrow E^{n+1} \cdots$$

(E^n in degree n). Hence, $LH^n(E)$ is the complex

$$0 \longrightarrow \mathrm{Coim} d_E^{n-1} \longrightarrow \mathrm{Ker} d_E^n \longrightarrow 0$$

($\mathrm{Ker} d_E^n$ *in degree 0).*

Proof. From the definition of the left t-structure on $\mathcal{D}(\mathcal{E})$ and Proposition 1.2.6, it is clear that $\tau^{\leq n}(E)$, $\tau^{\geq n}(E)$ are respectively canonically isomorphic to

$$\cdots \longrightarrow E^{n-2} \longrightarrow E^{n-1} \longrightarrow \mathrm{Ker} d_E^n \longrightarrow 0$$

($\mathrm{Ker} d_E^n$ in degree n) and to

$$0 \longrightarrow \mathrm{Ker} d_E^{n-1} \longrightarrow E^{n-1} \longrightarrow E^n \longrightarrow \cdots$$

(E^n in degree n). Hence, it is sufficient to prove that this last complex is isomorphic in $\mathcal{D}(\mathcal{E})$ to the complex:

$$0 \longrightarrow \mathrm{Coim} d_E^{n-1} \longrightarrow E^n \longrightarrow E^{n+1} \longrightarrow \cdots$$

(E^n in degree n) through the morphism u induced by the canonical morphism

$$j_E^{n-1} : E^{n-1} \longrightarrow \mathrm{Coim} d_E^{n-1}.$$

Since $\tau^{\geq n+1}(u)$ is clearly an isomorphism in $\mathcal{D}(\mathcal{E})$, it is sufficient to show that so is $\tau^{\leq n}(u)[n]$. This morphism is represented by the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker} d_E^{n-1} \arrow[r, "i_E^{n-1}"] \arrow[d, "0"] & E^{n-1} \arrow[r, "\delta_E^{n-1}"] \arrow[d, "j_E^{n-1}"] & \mathrm{Ker} d_E^n \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \mathrm{Coim} d_E^{n-1} \arrow[r, "\delta_E^{\prime\, n-1}"] & \mathrm{Ker} d_E^n \arrow[r] & 0
\end{tikzcd}$$

and its mapping cone is the complex

$$0 \longrightarrow \mathrm{Ker} d_E^{n-1} \xrightarrow{-i_E^{n-1}} E^{n-1} \xrightarrow{\begin{pmatrix} -\delta_E^{n-1} \\ j_E^{n-1} \end{pmatrix}} \mathrm{Ker} d_E^n \oplus \mathrm{Coim} d_E^{n-1} \xrightarrow{\begin{pmatrix} 1 & \delta_E'^{n-1} \end{pmatrix}} \mathrm{Ker} d_E^n \longrightarrow 0$$

($\mathrm{Ker} d_E^n$ in degree 0). This complex is clearly strictly exact in degree -3, -2 and 0. To show that it is strictly exact in degree -1, it is sufficient to note that

$$\mathrm{Coim} \begin{pmatrix} -\delta_E^{n-1} \\ j_E^{n-1} \end{pmatrix} \simeq \mathrm{Coim} d_E^{n-1}$$

and that

$$\mathrm{Coim} d_E^{n-1} \xrightarrow{\begin{pmatrix} -\delta_E'^{n-1} \\ 1 \end{pmatrix}} \mathrm{Ker} d_E^n \oplus \mathrm{Coim} d_E^{n-1}$$

is a kernel of

$$\mathrm{Ker} d_E^n \oplus \mathrm{Coim} d_E^{n-1} \xrightarrow{\begin{pmatrix} 1 & \delta_E'^{n-1} \end{pmatrix}} \mathrm{Ker} d_E^n.$$

$\square$

Corollary 1.2.19. *Let E be a complex of $\mathcal{E}$. Then,*

(a) *The cohomology object $LH^n(E)$ vanish if and only if the complex E is strictly exact in degree n.*

(b) *The complex E is an object of the category $\mathcal{D}^{\leq 0}(\mathcal{E})$ (resp. $\mathcal{D}^{\geq 0}(\mathcal{E})$) associated to the left t-structure of $\mathcal{D}(\mathcal{E})$ if and only if E is strictly exact in each strictly positive (resp. negative) degree.*

Corollary 1.2.20. *The left heart of $\mathcal{E}$ is equivalent to the localization of the full subcategory of $\mathcal{K}(\mathcal{E})$ consisting of complexes E of the form*

$$0 \longrightarrow E_1 \xrightarrow{\delta_E} E_0 \longrightarrow 0$$

(E_0 in degree 0, δ_E monomorphism) by the multiplicative system formed by morphisms $u : E \longrightarrow F$ such that the square

$$\begin{tikzcd} F_1 \arrow[r, "\delta_F"] & F_0 \\ E_1 \arrow[u, "u_1"] \arrow[r, "\delta_E"'] & E_0 \arrow[u, "u_0"'] \end{tikzcd}$$

is both cartesian and cocartesian.

Definition 1.2.21. An object A of $\mathcal{LH}(\mathcal{E})$ *is represented by the monomorphism* $f : E \longrightarrow F$ if it is isomorphic to the complex

$$0 \longrightarrow E \xrightarrow{f} F \longrightarrow 0$$

where F is in degree 0.

Remark 1.2.22. Through the canonical equivalence

$$\mathcal{D}(\mathcal{E})^{\mathrm{op}} \approx \mathcal{D}(\mathcal{E}^{\mathrm{op}}),$$

the left t-structure of $\mathcal{D}(\mathcal{E}^{\mathrm{op}})$ gives a second t-structure on $\mathcal{D}(\mathcal{E})$. We call it the *right t-structure of* $\mathcal{D}(\mathcal{E})$. We denote by $\mathcal{RH}(\mathcal{E})$ its heart and by RH^n the corresponding cohomology functors. The reader will easily dualize the preceding discussion and make the link between the right t-structure of $\mathcal{D}(\mathcal{E})$ and strictly coexact sequences of $\mathcal{E}$.

1.2.3 The canonical embedding of $\mathcal{E}$ in $\mathcal{LH}(\mathcal{E})$

In this subsection, we study the canonical embedding

$$\mathcal{E} \longrightarrow \mathcal{LH}(\mathcal{E})$$

and we show that it induces an equivalence at the level of derived categories.

Definition 1.2.23. We denote

$$I : \mathcal{E} \longrightarrow \mathcal{LH}(\mathcal{E})$$

the canonical functor which sends an object E of $\mathcal{E}$ to the complex

$$0 \longrightarrow E \longrightarrow 0$$

(E in degree 0) viewed as an object of $\mathcal{LH}(\mathcal{E})$.

Lemma 1.2.24. *Assume the square*

$$\begin{tikzcd} F_1 \arrow[r, "\delta_F"] & F_0 \\ E_1 \arrow[u, "u_1"] \arrow[r, "\delta_E"'] & E_0 \arrow[u, "u_0"'] \end{tikzcd}$$

is cocartesian. Then $\mathrm{Coker}\delta_E \simeq \mathrm{Coker}\delta_F$.

Proof. Denote by

$$e_F : F_0 \longrightarrow \mathrm{Coker}\delta_F$$

the canonical epimorphism. It is sufficient to show that

$$e_F \circ u_0 : E_0 \longrightarrow \mathrm{Coker}\delta_F$$

is a cokernel of δ_E. Assume $x : E_0 \longrightarrow X$ is such that $x \circ \delta_E = 0$. It follows from our assumptions that there is $x' : F_0 \longrightarrow X$ such that $x' \circ \delta_F = 0$ and $x' \circ u_0 = f$. Since e_F is a cokernel of δ_F, there is $x'' : \mathrm{Coker}\delta_F \longrightarrow X$ such that $x'' \circ e_F = x'$. Therefore $x'' \circ (e_F \circ u_0) = x$ and x'' is clearly the only morphism satisfying this equality. $\square$

Definition 1.2.25. Thanks to Corollary 1.2.20, the preceding lemma allows us to define a functor

$$C : \mathcal{LH}(\mathcal{E}) \rightarrow \mathcal{E}$$

by sending an object A represented by the monomorphism

$$E_1 \xrightarrow{\delta_E} E_0$$

to $\mathrm{Coker}\delta_E$. For any object A of $\mathcal{LH}(\mathcal{E})$, we call $C(A)$ the *classical part* of A.

Proposition 1.2.26. *We have a canonical isomorphism*

$$i : C \circ I \xrightarrow{\sim} \mathrm{id}_{\mathcal{E}}$$

and a canonical epimorphism

$$e : \mathrm{id}_{\mathcal{LH}(\mathcal{E})} \longrightarrow I \circ C.$$

Together, they induce the adjunction isomorphism

$$\mathrm{Hom}_{\mathcal{LH}(\mathcal{E})}(A, I(E)) = \mathrm{Hom}_{\mathcal{E}}(C(A), E).$$

In particular, $\mathcal{E}$ is a reflective subcategory of $\mathcal{LH}(\mathcal{E})$.

Proof. Let E be an object of $\mathcal{E}$. Since the cokernel of

$$0 \rightarrow E$$

is clearly isomorphic to E, we get a canonical isomorphism

$$i(E) : C \circ I(E) \rightarrow E.$$

Let A be an object of $\mathcal{LH}(\mathcal{E})$ represented by the monomorphism

$$E_1 \xrightarrow{\delta_E} E_0.$$

The canonical morphism

$$E_0 \longrightarrow \mathrm{Coker}\delta_E$$

induces a morphism

$$e(A) : A \longrightarrow I \circ C(A).$$

Since the square

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Coker}\delta_E \\ E_1 \arrow[u] \arrow[r, "\delta_E"] & E_0 \arrow[u] \end{tikzcd}$$

is cocartesian, $e(A)$ is an epimorphism in $\mathcal{LH}(\mathcal{E})$.

From the general results on adjunction formulas, we know that it is sufficient to show that both the composition of

$$e(I(E)) : I(E) \longrightarrow I \circ C \circ I(E)$$

and

$$I(i(E)) : I \circ C \circ I(E) \longrightarrow I(E)$$

and the composition of

$$C(e(A)) : C(A) \longrightarrow C \circ I \circ C(A)$$

and

$$i(C(A)) : C \circ I \circ C(A) \longrightarrow C(A)$$

give identity morphisms. This follows obviously from the definition of e and i. □

Corollary 1.2.27. *The canonical functor*

$$I : \mathcal{E} \longrightarrow \mathcal{LH}(\mathcal{E})$$

is fully faithful. Moreover, a sequence

$$E' \longrightarrow E \longrightarrow E''$$

is strictly exact in $\mathcal{E}$ if and only if the sequence

$$I(E') \longrightarrow I(E) \longrightarrow I(E'')$$

is exact in $\mathcal{LH}(\mathcal{E})$.

Proof. From the preceding proposition, it follows that

$$\operatorname{Hom}_{\mathcal{LH}(\mathcal{E})}(I(E), I(F)) = \operatorname{Hom}_{\mathcal{E}}(C \circ I(E), F).$$

Since $C \circ I \simeq \mathrm{id}_{\mathcal{E}}$, we see that I is fully faithful.

Assume the sequence

$$0 \longrightarrow E' \xrightarrow{e'} E \xrightarrow{e''} E'' \longrightarrow 0$$

is strictly exact. By a well known property of the heart of a t-structure, the cokernel of $I(e') : I(E') \longrightarrow I(E'')$ is obtained by applying the functor LH^0 to the complex

$$0 \longrightarrow E' \xrightarrow{e'} E \longrightarrow 0$$

(E in degree 0). Since e' is a monomorphism, $\mathrm{Coker} I(e')$ is represented by this same complex. The map

$$\begin{tikzcd} 0 \arrow[r] & E' \arrow[r, "e'"] \arrow[d, "0"] & E \arrow[r] \arrow[d, "e''"] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & E'' \arrow[r] & 0 \end{tikzcd}$$

being clearly a strict quasi-isomorphism, it follows that $\mathrm{Coker} I(e') \simeq I(E'')$. From the adjunction formula of Proposition 1.2.26, we know that I is kernel preserving. Hence, the sequence

$$0 \longrightarrow I(E') \xrightarrow{I(e')} I(E) \xrightarrow{I(e'')} I(E'') \longrightarrow 0$$

is exact.

Assume now that the sequence

$$E' \xrightarrow{e'} E \xrightarrow{e''} E''$$

is strictly exact. Applying the preceding result to the sequence

$$0 \longrightarrow \mathrm{Ker} e' \longrightarrow E \longrightarrow \mathrm{Ker} e'' \longrightarrow 0$$

and using the fact that I is kernel preserving we see easily that the sequence

$$I(E') \longrightarrow I(E) \longrightarrow I(E'')$$

is exact.

Finally, assume the sequence

$$I(E') \xrightarrow{I(e')} I(E) \xrightarrow{I(e'')} I(E'')$$

is exact. From what precedes, it follows that

$$I(\mathrm{Coim} e') = \mathrm{Coim} I(e'),$$
$$I(\mathrm{Ker} e'') = \mathrm{Ker} I(e'').$$

Since $\mathrm{Coim} I(e') \simeq \mathrm{Ker} I(e'')$, the result follows from the fact that I is fully faithful. □

Proposition 1.2.28.

(a) *An object A of $\mathcal{LH}(\mathcal{E})$ represented by the monomorphism*

$$E_1 \xrightarrow{\delta_E} E_0$$

is in the essential image of I if and only if δ_E is strict.

(b) *Assume*

$$A \longrightarrow B$$

is a monomorphism in $\mathcal{LH}(\mathcal{E})$ and B is in the essential image of I. Then A is also in the essential image of I.

(c) *Assume*

$$0 \longrightarrow A' \xrightarrow{a'} A \xrightarrow{a''} A'' \longrightarrow 0$$

is a short exact sequence in $\mathcal{LH}\,\mathcal{E})$ where both A' and A'' are in the essential image of I. Then A is also in the essential image of I.

Proof. (a) & (b) If δ_E is strict, then the sequence

$$0 \longrightarrow E_1 \xrightarrow{\delta_E} E_0 \longrightarrow \mathrm{Coker}\delta_E \longrightarrow 0$$

is strictly exact and by applying the functor I, we see that $A \simeq I(\mathrm{Coker}\delta_E)$.

Assume now that there is an object F of $\mathcal{E}$ and a monomorphism

$$A \longrightarrow I(F).$$

By Proposition 1.2.26, we know that this monomorphism is induced by a morphism

$$C(A) \longrightarrow F.$$

Hence, the canonical morphism

$$A \longrightarrow I(C(A))$$

is also a monomorphism. This means, by definition, that the complex

$$0 \longrightarrow E_1 \xrightarrow{\delta_E} E_0 \longrightarrow \mathrm{Coker}\delta_E \longrightarrow 0$$

is strictly exact at E_0. Therefore δ_E is strict and the conclusion follows.

(c) Assume A is represented by the monomorphism

$$E_1 \xrightarrow{\delta_E} E_0$$

and A'' is isomorphic to $I(E'')$ where E'' is an object of $\mathcal{E}$. Since the morphism $a'' : A \longrightarrow A''$ comes from a morphism $C(A) \longrightarrow E''$, it is represented by a morphism of complexes

$$\begin{tikzcd} E_1 \arrow[r, "\delta_E"] \arrow[d, "0"] & E_0 \arrow[d, "\alpha"] \\ 0 \arrow[r] & E'' \end{tikzcd}$$

Since the mapping cone of this morphism is the complex

$$0 \longrightarrow E_1 \xrightarrow{\delta_E} E_0 \xrightarrow{\alpha} E'' \longrightarrow 0,$$

the kernel of a'' is represented by the monomorphism

$$E_1 \xrightarrow{\beta} \mathrm{Ker}\alpha.$$

associated to δ_E. By assumption, this kernel is in the essential image of I. By (a), it follows that β is a strict monomorphism. Since the canonical monomorphism

$$\mathrm{Ker}\alpha \longrightarrow E_0$$

is also strict, Proposition 1.1.7 shows that δ_E itself is a strict monomorphism and (a) allows us to conclude. $\square$

Definition 1.2.29. For any object A of $\mathcal{LH}(\mathcal{E})$, we define the *vanishing part* of A to be the kernel $V(A)$ of the canonical epimorphism

$$e(A) : A \longrightarrow I \circ C(A).$$

Remark 1.2.30. For any object A of $\mathcal{LH}(\mathcal{E})$, $V(A)$ is represented by a bimorphism. Moreover, $V(A) \simeq 0$ if and only if A is in the essential image of I.

Proposition 1.2.31. *The canonical embedding*

$$I : \mathcal{E} \longrightarrow \mathcal{LH}(\mathcal{E})$$

induces an equivalence of categories

$$\mathcal{D}(I) : \mathcal{D}(\mathcal{E}) \to \mathcal{D}(\mathcal{LH}(\mathcal{E})).$$

which exchanges the left t-structure of $\mathcal{D}(\mathcal{E})$ with the usual t-structure of $\mathcal{D}(\mathcal{LH}(\mathcal{E}))$.

Proof. By Corollary 1.2.27, we know that I transforms a strictly exact complex of $\mathcal{E}$ in an exact complex of $\mathcal{LH}(\mathcal{E})$. Therefore, there is a unique functor $\mathcal{D}(I)$ making the diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r, "I"] \arrow[d, "Q_{\mathcal{E}}"] & \mathcal{LH}(\mathcal{E}) \arrow[d, "Q_{\mathcal{LH}(\mathcal{E})}"] \\ \mathcal{D}(\mathcal{E}) \arrow[r, "\mathcal{D}(I)"] & \mathcal{D}(\mathcal{LH}(\mathcal{E})) \end{tikzcd}$$

commutative.

Since any object A of $\mathcal{LH}(\mathcal{E})$ may be represented by a monomorphism

$$E_1 \xrightarrow{\delta_E} E_0.$$

of $\mathcal{E}$, it has a resolution of the form

$$0 \to I(E_1) \to I(E_0) \to A \to 0.$$

Hence, using Proposition 1.2.28, we may apply the dual of [5, Lemma 4.6] to the essential image of I considered as a subset of $\operatorname{Ob}\mathcal{LH}(\mathcal{E})$. This shows that for any complex A of $\mathcal{LH}(\mathcal{E})$ there is a complex E of $\mathcal{E}$ and a quasi-isomorphism

$$I(E) \to A.$$

Thanks to a well-known result on derived categories, the conclusion follows from the fact that a complex E of $\mathcal{E}$ is strictly exact in a specific degree if and only if $\mathcal{D}(I)(E)$ is exact in the same degree. □

1.2.4 The category $\mathcal{LH}(\mathcal{E})$ as an abelian envelope of $\mathcal{E}$

In this subsection, we show that $\mathcal{LH}(\mathcal{E})$ is in some sense an abelian envelope of $\mathcal{E}$. Although we have not stated explicitly the dual results for $\mathcal{RH}(\mathcal{E})$, we will use them freely.

Definition 1.2.32. Let $\mathcal{A}$ an abelian category. We denote by $\mathcal{R}ex(\mathcal{E}, \mathcal{A})$ (resp. $\mathcal{L}ex(\mathcal{E}, \mathcal{A})$) the *category of right (resp. left) exact functors from $\mathcal{E}$ to $\mathcal{A}$.*

Proposition 1.2.33. *For any abelian category $\mathcal{A}$, the inclusion functor*

$$I : \mathcal{E} \to \mathcal{LH}(\mathcal{E})$$

is strictly exact and induces an equivalence of categories

$$I' : \mathcal{R}ex(\mathcal{LH}(\mathcal{E}), \mathcal{A}) \to \mathcal{R}ex(\mathcal{E}, \mathcal{A}).$$

By this equivalence, exact functors correspond to strictly exact functors.

Proof. It follows from Corollary 1.2.27 that I is strictly exact. Hence I' is a well defined functor. Let us prove that it is essentially surjective. Let

$$F : \mathcal{E} \longrightarrow \mathcal{A}$$

be a right exact functor.

The functor

$$H^0 \circ \mathcal{K}(F) \circ \tau^{\leq 0} : \mathcal{K}(\mathcal{E}) \longrightarrow \mathcal{A}$$

transforms a quasi-isomorphism of $\mathcal{K}(\mathcal{E})$ into an isomorphism of $\mathcal{A}$. As a matter of fact, let

$$u : X \longrightarrow Y$$

be such a quasi-isomorphism. Since $Q(u)$ is an isomorphism in $\mathcal{D}(\mathcal{E})$, so is $\tau^{\leq 0}Q(u) \simeq Q(\tau^{\leq 0}u)$. Denote by Z the mapping cone of

$$\tau^{\leq 0}u : \tau^{\leq 0}X \longrightarrow \tau^{\leq 0}Y.$$

By construction, $Z^k = 0$ for $k > 0$. Since Z is strictly exact, the sequence

$$Z^{-2} \longrightarrow Z^{-1} \longrightarrow Z^0 \longrightarrow 0$$

is strictly exact. Applying F, we get the exact sequence

$$F(Z^{-2}) \longrightarrow F(Z^{-1}) \longrightarrow F(Z^0) \longrightarrow 0.$$

Hence $H^k(\mathcal{K}(F)(Z)) = 0$ for $k \geq -1$. Since the triangle

$$\mathcal{K}(F)(\tau^{\leq 0}X) \longrightarrow \mathcal{K}(F)(\tau^{\leq 0}Y) \longrightarrow \mathcal{K}(F)(Z) \xrightarrow{+1}$$

is distinguished in $\mathcal{K}(\mathcal{A})$, the long exact sequence of cohomology shows that

$$H^0 \circ \mathcal{K}(F) \circ \tau^{\leq 0}(u)$$

is an isomorphism in $\mathcal{A}$.

It follows from the preceding discussion that there is a functor

$$G : \mathcal{D}(\mathcal{E}) \longrightarrow \mathcal{A}$$

such that $G \circ Q = H^0 \circ \mathcal{K}(F) \circ \tau^{\leq 0}$. Let

$$F' : \mathcal{LH}(\mathcal{E}) \longrightarrow \mathcal{A}$$

be the restriction of G to $\mathcal{LH}(\mathcal{E})$. Since $F' \circ I \simeq F$, it remains to show that F' is right exact. Assume

$$0 \longrightarrow A' \xrightarrow{a'} A \xrightarrow{a''} A'' \longrightarrow 0$$

is an exact sequence in $\mathcal{LH}(\mathcal{E})$. Since we may replace A' and A by isomorphic objects, we may assume that $A'^k = A^k = 0$ for $k > 0$ and that a' is induced by a morphism of $\mathcal{K}(\mathcal{E})$ that we still denote a'. Let Z be the mapping cone of a'. By construction, $Z \in \mathcal{K}^{\leq 0}(\mathcal{E})$ and we have a distinguished triangle

$$A' \xrightarrow{a'} A \longrightarrow Z \xrightarrow{+1}$$

in $\mathcal{K}(\mathcal{E})$. Hence, $A'' \simeq Z$ in $\mathcal{D}(\mathcal{E})$ and

$$F'(A'') \simeq H^0(\mathcal{K}(F)(Z)).$$

Applying H^0 to the distinguished triangle

$$\mathcal{K}(F)(A') \longrightarrow \mathcal{K}(F)(A) \longrightarrow \mathcal{K}(F)(Z) \xrightarrow{+1}$$

of $\mathcal{K}(\mathcal{A})$, we get the exact sequence

$$F'(A') \longrightarrow F'(A) \longrightarrow F'(A'') \longrightarrow 0.$$

Note that when F is strictly exact, $\mathcal{K}(F)(Z)$ is strictly exact in any degree $k \neq 0$ and we get the short exact sequence

$$0 \longrightarrow F'(A') \longrightarrow F'(A) \longrightarrow F'(A'') \longrightarrow 0.$$

To see that I' is fully faithful, it is sufficient to recall that any object A of $\mathcal{LH}(\mathcal{E})$ may be embedded in an exact sequence

$$I(E_1) \longrightarrow I(E_0) \longrightarrow A \longrightarrow 0.$$

□

Lemma 1.2.34. *Let $\mathcal{E}$ be a full subcategory of the abelian category $\mathcal{A}$. Assume $\mathcal{E}$ is essentially stable by subobjects (i.e. for any monomorphism*

$$A \longrightarrow E$$

of $\mathcal{A}$ with E in $\mathcal{E}$ there is E' in $\mathcal{E}$ and an isomorphism $A \simeq E'$). Then,

(a) *Any morphism of $\mathcal{E}$ has a kernel and a coimage and they are computable in $\mathcal{A}$.*

(b) *A sequence*

$$E \xrightarrow{u} F \xrightarrow{v} G$$

of $\mathcal{E}$ is exact in $\mathcal{A}$ if and only if

$$\operatorname{Coim} u \simeq \operatorname{Ker} v$$

in $\mathcal{E}$.

(c) If

$$\begin{tikzcd} E \arrow[r, "u"] & F \\ E' \arrow[u] \arrow[r, "u'"'] & F' \arrow[u] \end{tikzcd}$$

is a cartesian square in $\mathcal{E}$ and u is a strict epimorphism then so is u'.

Proof. To avoid confusions, we will make use of the canonical inclusion functor $J : \mathcal{E} \longrightarrow \mathcal{A}$.

(a) Let

$$u : E \longrightarrow F$$

be a morphism of $\mathcal{E}$. Since

$$\mathrm{Ker} J(u) \longrightarrow J(E)$$

is a monomorphism, there is an object K of $\mathcal{E}$ and an isomorphism

$$\mathrm{Ker} J(u) \simeq J(K).$$

This gives us a morphism

$$J(K) \longrightarrow J(E)$$

which is a kernel of $J(u)$. Since $\mathcal{E}$ is a full subcategory of $\mathcal{A}$, this morphism is of the form $J(k)$ where

$$k : K \longrightarrow E$$

is a morphism in $\mathcal{E}$. One checks easily that k is a kernel of u in $\mathcal{E}$. Hence,

$$J(\mathrm{Ker} u) \simeq \mathrm{Ker} J(u).$$

Since the canonical morphism

$$\mathrm{Coim} J(u) \longrightarrow J(F)$$

is a monomorphism, there is an object C of $\mathcal{E}$ and an isomorphism

$$\mathrm{Coim} J(u) \simeq J(C).$$

Proceeding as above, we get a morphism

$$c : E \longrightarrow C$$

such that

$$J(c) : J(E) \longrightarrow J(C)$$

is a cokernel of

$$J(k) : J(\mathrm{Ker} u) \longrightarrow J(E).$$

Therefore,

$$c : E \to C$$

is a cokernel of $k : \mathrm{Ker} u \longrightarrow E$ and C is a coimage of u. Hence,

$$J(\mathrm{Coim} u) \simeq \mathrm{Coim} J(u).$$

Parts (b) and (c) follow directly from (a). □

Proposition 1.2.35. *Let $\mathcal{E}$ be a quasi-abelian category and let $\mathcal{A}$ be an abelian category. Assume that the functor*

$$J : \mathcal{E} \to \mathcal{A}$$

is fully faithful and that

(a) for any monomorphism

$$A \longrightarrow J(E)$$

of $\mathcal{A}$ there is an object E' of $\mathcal{E}$ and an isomorphism

$$A \simeq J(E'),$$

(b) for any object A of $\mathcal{A}$, there is an epimorphism

$$J(E) \longrightarrow A$$

where E is an object of $\mathcal{E}$.

Then, J extends to an equivalence of categories

$$\mathcal{LH}(\mathcal{E}) \approx \mathcal{A}.$$

Proof. It follows from (b) that for any complex $A \in \mathcal{D}^-(\mathcal{A})$ there is a complex E of $\mathcal{D}^-(\mathcal{E})$ and a quasi-isomorphism

$$J(E) \longrightarrow A.$$

Moreover, thanks to the preceding lemma, a complex $E \in \mathcal{D}^-(\mathcal{E})$ is strictly exact in degree k if and only if $J(E)$ is exact in degree k. It follows from these facts that J induces an equivalence

$$\mathcal{D}^-(\mathcal{E}) \longrightarrow \mathcal{D}^-(\mathcal{A})$$

which exchanges the left t-structure of $\mathcal{D}^-(\mathcal{E})$ with the canonical t-structure of $\mathcal{D}^-(\mathcal{A})$. In particular,

$$\mathcal{LH}(\mathcal{E}) \approx \mathcal{A}.$$

□

1.3 Derivation of quasi-abelian functors

In this section, we assume that $\mathcal{E}$ and $\mathcal{F}$ are quasi-abelian categories and we will give conditions for an additive functor

$$F : \mathcal{E} \to \mathcal{F}$$

to be left or right derivable (Although we do note state explicitely the corresponding results for multivariate functors, the reader will figure them out easily). We will also investigate to what extend F is determined by its left and right derived functors. Finally, we will explain how to replace F with a functor

$$G : \mathcal{LH}(\mathcal{E}) \to \mathcal{LH}(\mathcal{E})$$

with the same left or right derived functor.

1.3.1 Derivable and explicitly derivable quasi-abelian functors

As in the abelian case, we introduce the following definition.

Definition 1.3.1. Let

$$F : \mathcal{E} \to \mathcal{F}$$

be an additive functor and denote as usual

$$Q_{\mathcal{E}} : \mathcal{K}(\mathcal{E}) \to \mathcal{D}(\mathcal{E}), \quad Q_{\mathcal{F}} : \mathcal{K}(\mathcal{F}) \to \mathcal{D}(\mathcal{F})$$

the canonical functors.

Assume we are given a triangulated functor

$$G : \mathcal{D}^+(\mathcal{E}) \to \mathcal{D}^+(\mathcal{F})$$

and a morphism

$$g : Q_{\mathcal{F}} \circ \mathcal{K}^+(F) \to G \circ Q_{\mathcal{E}}.$$

Then, (G, g) is *a right derived functor of* F if for any other such pair (G', g'), there is a unique morphism

$$h : G \to G'$$

making the diagram

$$\begin{tikzcd} & G \circ Q_{\mathcal{E}} \arrow[dd, "h \circ Q_{\mathcal{E}}"] \\ Q_{\mathcal{F}} \circ \mathcal{K}^+(F) \arrow[ur, "g"] \arrow[dr, "g'"'] & \\ & G' \circ Q_{\mathcal{E}} \end{tikzcd}$$

commutative. The functor F is *right derivable* if it has a right derived functor. In this case, since two right derived functors of F are canonically isomorphic, we may select a specific one. We denote such a functor RF and call it *the right derived functor of* F.

Dually, assume we are given a triangulated functor

$$G : \mathcal{D}^-(\mathcal{E}) \longrightarrow \mathcal{D}^-(\mathcal{F})$$

and a morphism

$$g : G \circ Q_{\mathcal{E}} \longrightarrow Q_{\mathcal{F}} \circ \mathcal{K}^-(F).$$

Then, (G, g) is *a left derived functor of* F if, for any other such pair (G', g'), there is a unique morphism

$$h : G' \longrightarrow G$$

making the diagram

$$\begin{tikzcd} G' \circ Q_{\mathcal{E}} \arrow[dd, "h \circ Q_{\mathcal{E}}"'] \arrow[dr, "g'"] & \\ & Q_{\mathcal{F}} \circ \mathcal{K}^-(F) \\ G \circ Q_{\mathcal{E}} \arrow[ur, "g"'] & \end{tikzcd}$$

commutative. The functor F is *left derivable* if it has a left derived functor. In this case, since two left derived functors of F are canonically isomorphic, we may select a specific one. We denote such a functor LF and call it *the left derived functor of* F.

In order to give a criterion for derivability, we will adapt the usual results for abelian functors.

Definition 1.3.2. Let

$$F : \mathcal{E} \longrightarrow \mathcal{F}$$

be an additive functor.

A full additive subcategory $\mathcal{P}$ of $\mathcal{E}$ is *F-projective* if

(a) for any object E of $\mathcal{E}$ there is an object P of $\mathcal{P}$ and a strict epimorphism

$$P \longrightarrow E.$$

(b) in any strictly exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

of $\mathcal{E}$ where E and E'' are object of $\mathcal{P}$, E' is also in $\mathcal{P}$.

(c) for any strictly exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

where E', E and E'' are objects of $\mathcal{P}$, the sequence

$$0 \longrightarrow F(E') \longrightarrow F(E) \longrightarrow F(E'') \longrightarrow 0$$

is strictly exact in $\mathcal{F}$.

Dually, a full additive subcategory $\mathcal{I}$ of $\mathcal{E}$ is F-*injective* if

(a) for any object E of $\mathcal{E}$ there is an object I of $\mathcal{I}$ and a strict monomorphism

$$E \longrightarrow I.$$

(b) in any strictly exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

of $\mathcal{E}$ where E' and E are object of $\mathcal{I}$, E'' is also in $\mathcal{I}$.

(c) for any strictly exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

where E', E and E'' are objects of $\mathcal{I}$, the sequence

$$0 \longrightarrow F(E') \longrightarrow F(E) \longrightarrow F(E'') \longrightarrow 0$$

is strictly exact in $\mathcal{F}$.

Lemma 1.3.3. *Let $\mathcal{P}$ be a subset of $\mathrm{Ob}(\mathcal{E})$. Assume that, for any object E of $\mathcal{E}$, there is an object $P \in \mathcal{P}$ and a strict epimorphism*

$$P \longrightarrow E.$$

Then, for any object E of $\mathcal{C}^-(\mathcal{E})$ there is an object P of $\mathcal{C}^-(\mathcal{P})$ and a quasi-isomorphism

$$u : P \longrightarrow E$$

where each

$$u^k : P^k \longrightarrow E^k$$

is a strict epimorphism with $P^k \in \mathcal{P}$.

Proof. We may restrict ourselves to the case where $E^k = 0$ for $k > 0$. To simplify the notations, we set as usual $E_k = E^{-k}$. We will proceed by induction. Assume we have already P_k, u_k, d_k^P such that

$$\begin{tikzcd} 0 \arrow[r] & E_k \arrow[r, "d_k^E"] & E_{k-1} \arrow[r, dotted, dash] & E_0 \arrow[r] & 0 \\ 0 \arrow[r] & P_k \arrow[r, "d_k^P"] \arrow[u, "u_k"] & P_{k-1} \arrow[r, dotted, dash] \arrow[u, "u_{k-1}"] & P_0 \arrow[r] \arrow[u, "u_0"] & 0 \end{tikzcd}$$

is a k-quasi-isomorphism (i.e. the mapping cone is strictly exact in degree greater or equal to k). Let us form the cartesian square

$$\begin{tikzcd} E_{k+1} \arrow[r, "d_{k+1}^E"] & \mathrm{Ker} d_k^E \\ E'_{k+1} \arrow[r, "v"] \arrow[u, "v'"] & \mathrm{Ker} d_k^P \arrow[u, "u_k"] \end{tikzcd}$$

Let

$$w : P_{k+1} \longrightarrow E'_{k+1}$$

be a strict epimorphism with P_{k+1} in $\mathcal{P}$ and set

$$d_{k+1}^P = v \circ w \qquad u_{k+1} = v' \circ w.$$

Let us show that

$$\begin{tikzcd} 0 \arrow[r] & E_{k+1} \arrow[r, "d_{k+1}^E"] & E_k \arrow[r, dotted, dash] & E_0 \arrow[r] & 0 \\ 0 \arrow[r] & P_{k+1} \arrow[r, "d_{k+1}^P"] \arrow[u, "u_{k+1}"] & P_k \arrow[r, dotted, dash] \arrow[u, "u_k"] & P_0 \arrow[r] \arrow[u, "u_0"] & 0 \end{tikzcd}$$

is a $(k+1)$-quasi-isomorphism. It follows from the definition of the mapping cone and from the induction hypothesis that the only thing to prove is that the sequence

$$P_{k+1} \xrightarrow{\begin{pmatrix} u_{k+1} \\ -d_{k+1}^P \end{pmatrix}} E_{k+1} \oplus P_k \xrightarrow{\begin{pmatrix} d_{k+1}^E & u_k \\ 0 & -d_k^P \end{pmatrix}} E_k \oplus P_{k-1}$$

is strictly exact. By construction,

$$E'_{k+1} \xrightarrow{\begin{pmatrix} v' \\ -v \end{pmatrix}} E_{k+1} \oplus P_k$$

is a kernel of

$$E_{k+1} \oplus P_k \xrightarrow{\begin{pmatrix} d_{k+1}^E & u_k \\ 0 & -d_k^P \end{pmatrix}} E_k \oplus P_{k-1}$$

Since w is a strict epimorphism, the conclusion follows easily.

To conclude, let us show that u_{k+1} is a strict epimorphism. By applying LH_k, it follows from the induction hypothesis that

$$u_k : \mathrm{Ker} d_k^P \longrightarrow \mathrm{Ker} d_k^E$$

is a strict epimorphism. Therefore, v' is also a strict epimorphism and by composition, so is u_{k+1}. $\square$

Proposition 1.3.4. *Let $\mathcal{P}$ be an F-projective subcategory of $\mathcal{E}$. Then, the full subcategory $\mathcal{N}^-(\mathcal{P})$ of $\mathcal{K}^-(\mathcal{P})$ formed by strictly exact complexes is a null system and the canonical functor*

$$\mathcal{K}^-(\mathcal{P})/\mathcal{N}^-(\mathcal{P}) \to \mathcal{D}^-(\mathcal{E})$$

is an equivalence of categories. Dually, let $\mathcal{I}$ be an F-injective subcategory of $\mathcal{E}$. Then, the full subcategory $\mathcal{N}^+(\mathcal{I})$ of $\mathcal{K}^+(\mathcal{I})$ formed by strictly exact complexes is a null system and the canonical functor

$$\mathcal{K}^+(\mathcal{I})/\mathcal{N}^+(\mathcal{I}) \to \mathcal{D}^+(\mathcal{E})$$

is an equivalence of categories.

Proof. Thanks to Lemma 1.3.3, the proof goes as in the abelian case. $\square$

Proposition 1.3.5. *Let $\mathcal{E}$, $\mathcal{F}$ be quasi-abelian categories and let*

$$F : \mathcal{E} \to \mathcal{F}$$

be an additive functor.

(a) *Assume $\mathcal{E}$ has an F-injective subcategory. Then, F has a right derived functor*

$$RF : \mathcal{D}^+(\mathcal{E}) \to \mathcal{D}^+(\mathcal{F}).$$

(b) *Dually, assume $\mathcal{E}$ has an F-projective subcategory. Then, F has a left derived functor*

$$LF : \mathcal{D}^-(\mathcal{E}) \to \mathcal{D}^-(\mathcal{F}).$$

Proof. Thanks to Lemma 1.3.3 and Proposition 1.3.4, the proof goes as in the abelian case. $\square$

The preceding proposition is the main tool to show that a functor is derivable. However, it does not give a necessary and sufficient condition for derivability. This is the reason of the following definition.

Definition 1.3.6. An additive functor

$$F : \mathcal{E} \longrightarrow \mathcal{F}$$

is *explicitly right (resp. left) derivable* if $\mathcal{E}$ has an F-injective (resp. F-projective) subcategory.

Remark 1.3.7. Let

$$F : \mathcal{E} \longrightarrow \mathcal{F}$$

be a right derivable left exact functor. Call F-acyclic an object I of $\mathcal{E}$ such that

$$F(I) \simeq RF(I)$$

and assume that for any object E of $\mathcal{E}$ there is an F-acyclic object I and a monomorphism

$$E \longrightarrow I.$$

Then, F-acyclic objects of $\mathcal{E}$ form an F-injective subcategory and F is explicitly right derivable.

1.3.2 Exactness properties of derived functors

Proposition 1.3.8. *Let*

$$F : \mathcal{E} \longrightarrow \mathcal{F}$$

be an additive functor of quasi-abelian categories and let $\mathcal{I}$ be an F-injective subcategory of $\mathcal{E}$. Consider the right derived functor

$$RF : \mathcal{D}^+(\mathcal{E}) \longrightarrow \mathcal{D}^+(\mathcal{F}).$$

Then

(LL) The functor RF is left exact for the left t-structures of $\mathcal{D}^+(\mathcal{E})$ and $\mathcal{D}^+(\mathcal{F})$ if and only if the image by F of any monomorphism

$$I_1 \longrightarrow I_0$$

of $\mathcal{E}$ where I_0, I_1 are objects of $\mathcal{I}$ is a monomorphism of $\mathcal{F}$.

(LR) The functor RF is left exact for the left t-structure of $\mathcal{D}^+(\mathcal{E})$ and the right t-structure of $\mathcal{D}^+(\mathcal{F})$ if and only if the image by F of any monomorphism

$$I_1 \longrightarrow I_0$$

of $\mathcal{E}$ where I_0, I_1 are objects of $\mathcal{I}$ is a strict monomorphism of $\mathcal{F}$.

(RL) The functor RF is left exact for the right t-structure of $\mathcal{D}^+(\mathcal{E})$ and the left t-structure $\mathcal{D}^+(\mathcal{F})$.

(RR) The functor RF is left exact for the right t-structure of $\mathcal{D}^+(\mathcal{E})$ and the right t-structure $\mathcal{D}^+(\mathcal{F})$.

Proof.
(LL) The condition is necessary. Let A be an object of $\mathcal{LH}(\mathcal{E})$ represented by a monomorphism

$$I_1 \xrightarrow{\delta} I_0$$

where I_0, I_1 are objects of $\mathcal{I}$. It follows that $RF(A)$ is isomorphic to the complex

$$0 \longrightarrow F(I_1) \xrightarrow{F(\delta)} F(I_0) \longrightarrow 0$$

($F(I_0)$ in degree 0). Our assumption implies that this complex is strictly exact in degree -1. Therefore, $F(\delta)$ is a monomorphism in $\mathcal{F}$.

The condition is also sufficient. Let E be an object of $\mathcal{D}^{\geq 0}(\mathcal{E})$. Since

$$E \simeq \tau^{\geq 0} E,$$

we may assume that $E^k = 0$ for $k < -1$. Replacing E by an isomorphic complex if necessary, we may even assume that E^k is an object of $\mathcal{I}$ for any $k \in \mathbb{Z}$. Since E is an object of $\mathcal{D}^{\geq 0}(\mathcal{E})$,

$$E^{-1} \longrightarrow E^0$$

is a monomorphism of $\mathcal{E}$. Hence,

$$F(E^{-1}) \longrightarrow F(E^0)$$

is a monomorphism of $\mathcal{F}$ and

$$RF(E) \simeq F(E)$$

is an object of $\mathcal{D}^{\geq 0}(\mathcal{F})$.
(LR) Let us show that the condition is necessary. Let A be an object of $\mathcal{LH}(\mathcal{E})$ represented by a monomorphism

$$I_1 \xrightarrow{\delta} I_0$$

where I_0, I_1 are objects of $\mathcal{I}$. It follows that $RF(A)$ is isomorphic to the complex

$$0 \longrightarrow F(I_1) \xrightarrow{F(\delta)} F(I_0) \longrightarrow 0$$

($F(I_0)$ in degree 0). Our assumption implies that this complex is strictly coexact in degree -1. Therefore, $F(\delta)$ is a strict monomorphism in $\mathcal{F}$.

The condition is also sufficient. Let E be an object of $\mathcal{D}^+(\mathcal{E})$ which is strictly exact in each strictly negative degree. Replacing E by an isomorphic complex if necessary, we may assume that $E^k = 0$ for $k < -1$ and that E^k is an object of $\mathcal{I}$ for any $k \geq -1$. Since E is strictly exact in degree -1, the differential

$$E^{-1} \to E^0$$

is a monomorphism. Therefore, our hypothesis shows that

$$F(E^{-1}) \to F(E^0)$$

is a strict monomorphism in $\mathcal{F}$ and the complex

$$RF(E) \simeq F(E)$$

is strictly coexact in each strictly negative degree.
(RL) & (RR) Let E be an object of $\mathcal{D}^+(\mathcal{E})$ which is strictly coexact in each strictly negative degree. Replacing E by an isomorphic complex if necessary, we may assume that $E^k = 0$ for $k < 0$ and that E^k is an object of $\mathcal{I}$ for any $k \geq 0$. Therefore, the complex

$$RF(E) \simeq F(E)$$

is strictly coexact in each strictly negative degree. □

Remark 1.3.9. One checks easily that the condition in part (LL) of the preceding proposition is equivalent to the fact that

$$LH^{-1} \circ RF(A) \simeq 0$$

for any object A of $\mathcal{LH}(\mathcal{E})$. Similarly the condition in part (LR) of the preceding proposition is equivalent to the fact that

$$RH^{-1} \circ RF(A) \simeq 0$$

for any object A of $\mathcal{LH}(\mathcal{E})$.

Proposition 1.3.10. *Let*

$$F : \mathcal{E} \to \mathcal{F}$$

be an explicitly right derivable functor of quasi-abelian categories and consider its right derived functor

$$RF : \mathcal{D}^+(\mathcal{E}) \to \mathcal{D}^+(\mathcal{F}).$$

Then the canonical morphism

$$I \circ F \longrightarrow LH^0 \circ RF \circ I \qquad (\textit{resp. } I \circ F \longrightarrow RH^0 \circ RF \circ I)$$

is an isomorphism if and only if F is RL (resp. RR) left exact.

Proof. We consider only the RL case, since the RR case may be treated similarly. Assume that

$$I \circ F \simeq LH^0 \circ RF \circ I.$$

Let

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

be a strict exact sequence of $\mathcal{E}$ and consider the induced distinguished triangle

$$I(E') \longrightarrow I(E) \longrightarrow I(E'') \xrightarrow{+1}$$

of $\mathcal{D}^+(E)$. Applying RF and passing to cohomology, we get the exact sequence

$$0 \longrightarrow LH^0 \circ RF \circ I(E') \longrightarrow LH^0 \circ RF \circ I(E) \longrightarrow LH^0 \circ RF \circ I(E'')$$

of $\mathcal{LH}(\mathcal{F})$. Using our assumption, we see that the sequence

$$0 \longrightarrow I \circ F(E') \longrightarrow I \circ F(E) \longrightarrow I \circ F(E'')$$

is exact in $\mathcal{LH}(\mathcal{F})$. Therefore the sequence,

$$0 \longrightarrow F(E') \longrightarrow F(E) \longrightarrow F(E'')$$

is strictly exact in $\mathcal{F}$ and F is RL left exact.

Conversely, assume F is RL left exact. Let $\mathcal{I}$ be an F-injective subcategory of $\mathcal{E}$ and let I be a resolution of an object E of $\mathcal{E}$ by objects of $\mathcal{I}$. The sequence

$$0 \longrightarrow E \longrightarrow I^0 \longrightarrow I^1$$

being strictly coexact in $\mathcal{E}$, our assumption shows that the sequence

$$0 \longrightarrow F(E) \longrightarrow F(I^0) \longrightarrow F(I^1)$$

is strictly exact in $\mathcal{F}$. Therefore,

$$I \circ F(E) \simeq LH^0 \circ F(I) \simeq LH^0 \circ RF \circ I(E)$$

as requested. $\square$

Proposition 1.3.11. *Let*

$$F : \mathcal{E} \longrightarrow \mathcal{F}$$

be an explicitly right derivable functor of quasi-abelian categories and consider its right derived functor

$$RF : \mathcal{D}^+(\mathcal{E}) \longrightarrow \mathcal{D}^+(\mathcal{F}).$$

Then

(LL) The functor RF is left exact for the left t-structures of $\mathcal{D}^+(\mathcal{E})$ and $\mathcal{D}^+(\mathcal{F})$ and

$$LH^0 \circ RF \circ I \simeq I \circ F$$

if and only if F is LL left exact.

(LR) The functor RF is left exact for the left t-structure of $\mathcal{D}^+(\mathcal{E})$ and the right t-structure of $\mathcal{D}^+(\mathcal{F})$ and

$$RH^0 \circ RF \circ I \simeq I \circ F$$

if and only if F is LR left exact.

(RL) The functor RF is left exact for the right t-structure of $\mathcal{D}^+(\mathcal{E})$ and the left t-structure $\mathcal{D}^+(\mathcal{F})$ and

$$LH^0 \circ RF \circ I \simeq I \circ F$$

if and only if F is RL left exact.

(RR) The functor RF is left exact for the right t-structure of $\mathcal{D}^+(\mathcal{E})$ and the right t-structure $\mathcal{D}^+(\mathcal{F})$ and

$$RH^0 \circ RF \circ I \simeq I \circ F$$

if and only if F is RR left exact.

Proof. (LL) This follows from the preceding proposition and Proposition 1.3.8.

(LR) From the preceding proposition and Proposition 1.1.15, the condition is clearly sufficient. Let us show that it is also necessary. By Proposition 1.3.10, we already know that F is RR left exact. Since F transforms any strict morphism into a strict morphism, to conclude, it is sufficient to show that F transforms any monomorphism into a strict monomorphism. Let A be the object of $\mathcal{LH}(\mathcal{E})$ represented by a monomorphism

$$E_1 \xrightarrow{\delta_E} E_0$$

of $\mathcal{E}$. Consider the associated distinguished triangle

$$E_1 \longrightarrow E_0 \longrightarrow A \xrightarrow{+1}$$

of $\mathcal{D}^+(\mathcal{E})$. Applying the functor RF and taking cohomology, we get the exact sequence

$$0 \longrightarrow RH^0 \circ RF \circ I(E_1) \longrightarrow RH^0 \circ RF \circ I(E_0) \longrightarrow RH^0 \circ RF(A)$$

of $\mathcal{RH}(\mathcal{F})$. Hence the sequence

$$0 \longrightarrow F(E_1) \xrightarrow{F(\delta_E)} F(E_0)$$

is strictly coexact in $\mathcal{F}$ and $F(\delta_E)$ is a strict monomorphism in $\mathcal{F}$.
(RL) & (RR) This follows directly from Proposition 1.3.10. $\square$

1.3.3 Abelian substitutes of quasi-abelian functors

In this subsection, our aim is to show that, under suitable conditions, a functor $F : \mathcal{E} \longrightarrow \mathcal{F}$ gives rise to a functor $G : \mathcal{LH}(\mathcal{E}) \longrightarrow \mathcal{LH}(\mathcal{F})$ which has the same left or right derived functor.

Definition 1.3.12. Two explicitly right (resp. left) derivable quasi-abelian functors are right (resp. left) equivalent if their right (resp. left) derived functors are isomorphic.

Proposition 1.3.13. *Let*

$$F : \mathcal{E} \longrightarrow \mathcal{F}$$

be an additive functor.

Assume F is explicitly right derivable. Then F is right equivalent to an explicitly right derivable left exact functor

$$F^0 : \mathcal{E} \longrightarrow \mathcal{F}$$

which is unique up to isomorphism.

Dually, assume F is explicitly left derivable. Then F is left equivalent to an explicitly left derivable right exact functor

$$F_0 : \mathcal{E} \longrightarrow \mathcal{F}$$

which is unique up to isomorphism.

Proof. Let $\mathcal{I}$ be an F-injective subcategory of $\mathcal{E}$. Let E be an object of $\mathcal{E}$ and let I be a right resolution of E by objects of $\mathcal{I}$. Since

$$RF(E) \simeq F(I),$$

it is clear that
$$LH^0 \circ RF(E)$$
is in the essential image of
$$I : \mathcal{E} \to \mathcal{LH}(\mathcal{E}).$$
Set
$$F^0 = C \circ LH^0 \circ I.$$
By construction, the functor F^0 is left exact and is isomorphic with F on $\mathcal{I}$. Therefore, $\mathcal{I}$ forms an F^0-injective subcategory of $\mathcal{E}$ and we get
$$RF^0 \simeq RF.$$
Hence F^0 is right equivalent to F.

Assume now that $G : \mathcal{E} \longrightarrow \mathcal{F}$ is an explicitly right derivable left exact functor which is right equivalent to F. Since we have
$$LH^0 \circ RG(E) \simeq I \circ G(E)$$
and $RF \simeq RG$, we see that
$$G(E) \simeq C \circ LH^0 \circ RG(E) \simeq F^0$$
and the conclusion follows. □

Proposition 1.3.14. *Let*
$$F : \mathcal{E} \longrightarrow \mathcal{F}$$
be an additive functor between quasi-abelian categories. Assume F is explicitly right derivable. Denote $\mathcal{I}$ an F-injective subcategory of $\mathcal{E}$ and consider the right derived functor
$$RF : \mathcal{D}^+(\mathcal{E}) \longrightarrow \mathcal{D}^+(\mathcal{F}).$$
In order that there exists an explicitly right derivable functor
$$G : \mathcal{LH}(\mathcal{E}) \longrightarrow \mathcal{LH}(\mathcal{F})$$
and an isomorphism
$$RF = RG \circ \mathcal{D}(I),$$
it is necessary and sufficient that one of the following equivalent conditions is satisfied:

(a) *The functor RF is left exact with respect to the left t-structures of $\mathcal{D}^+(\mathcal{E})$ and $\mathcal{D}^+(\mathcal{F})$.*

(b) The functor F^0 is strongly left exact.

In such a case, G is right equivalent to the explicitly right derivable left exact functor

$$LH^0 \circ RF : \mathcal{LH}(\mathcal{E}) \to \mathcal{LH}(\mathcal{F}).$$

Moreover, the restriction of this functor to $\mathcal{E}$ is isomorphic to F if and only if F is strongly left exact.

Proof. First, let us show that conditions (a) and (b) are equivalent.
(a) ⇒ (b). Let

$$0 \to A' \to A \to A'' \to 0$$

be a short exact sequence of $\mathcal{LH}(\mathcal{E})$. From the associated distinguished triangle

$$RF(A') \to RF(A) \to RF(A'') \xrightarrow{+1}$$

and the fact that $LH^{-1} \circ RF(A'') \simeq 0$, we deduce that the sequence

$$0 \to LH^0 \circ RF(A') \to LH^0 \circ RF(A) \to LH^0 \circ RF(A'')$$

is exact in $\mathcal{LH}(\mathcal{F})$. Hence,

$$LH^0 \circ RF : \mathcal{LH}(\mathcal{E}) \to \mathcal{LH}(\mathcal{F})$$

is a left exact functor. Consider now a morphism

$$e' : E' \to E$$

in $\mathcal{E}$. Since the sequence

$$0 \to \mathrm{Ker} e' \to E' \to E$$

is strictly exact in $\mathcal{E}$, it gives rise to an exact sequence in $\mathcal{LH}(\mathcal{E})$. Applying $LH^0 \circ RF$, we get the exact sequence

$$0 \to LH^0 \circ RF \circ I(\mathrm{Ker} e') \to LH^0 \circ RF \circ I(E') \to LH^0 \circ RF \circ I(E)$$

of $\mathcal{LH}(\mathcal{F})$. Therefore, the sequence

$$0 \to F^0(\mathrm{Ker} e') \to F^0(E') \to F^0(E)$$

is strictly exact in $\mathcal{F}$ and F^0 is a strongly left exact functor.
(b) ⇒ (a). Let

$$I_1 \to I_0$$

be a monomorphism of $\mathcal{E}$ with both I_1 and I_0 in $\mathcal{I}$. Since F^0 is strongly left exact,

$$F^0(I_1) \longrightarrow F^0(I_0)$$

is a monomorphism of $\mathcal{F}$. The conclusion follows from the fact that F^0 and F coincide on $\mathcal{I}$.

Now, let us come back to the main proof.

Necessity. Since RG is left exact with respect to the left t-structures of $\mathcal{D}^+(\mathcal{E})$ and $\mathcal{D}^+(\mathcal{F})$, so is RF and condition (a) is satisfied.

Sufficiency. Denote G the functor

$$LH^0 \circ RF : \mathcal{LH}(\mathcal{E}) \longrightarrow \mathcal{LH}(\mathcal{F})$$

and let $\mathcal{J}$ denote the full additive subcategory of $\mathcal{LH}(\mathcal{E})$ formed by the objects A such that

$$LH^k \circ RF(A) \simeq 0$$

for any $k \neq 0$. It follows from condition (a) and from the long exact sequence of cohomology associated to F that G is a left exact functor and that for any short exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

where A' and A are objects of $\mathcal{J}$, A'' is also an object of $\mathcal{J}$ and the sequence

$$0 \longrightarrow G(A') \longrightarrow G(A) \longrightarrow G(A'') \longrightarrow 0$$

is exact in $\mathcal{LH}(\mathcal{F})$. Let A be an object of $\mathcal{LH}(\mathcal{E})$. As an object of $\mathcal{D}(\mathcal{E})$, A is isomorphic to a complex

$$0 \longrightarrow I^{-1} \xrightarrow{d^{-1}} I^0 \xrightarrow{d^0} I^1 \longrightarrow \cdots$$

of objects of $\mathcal{I}$. It follows that A is represented by the monomorphism

$$I^{-1} \longrightarrow \mathrm{Ker} d^0.$$

Since the square

$$\begin{tikzcd} I^{-1} \arrow[r] & I^0 \\ I^{-1} \arrow[u] \arrow[r] & \mathrm{Ker} d^0 \arrow[u] \end{tikzcd}$$

is cartesian, it represents a monomorphism

$$A \longrightarrow J$$

where J is the object of $\mathcal{LH}(\mathcal{E})$ represented by the monomorphism

$$I^{-1} \longrightarrow I^0.$$

Since the sequence

$$0 \longrightarrow I(I^{-1}) \longrightarrow I(I^0) \longrightarrow J \longrightarrow 0$$

is exact in $\mathcal{LH}(\mathcal{E})$ and $I(I^{-1})$, $I(I^0)$ are objects of $\mathcal{J}$, we see that J is also an object of $\mathcal{J}$. Together with what precedes, this shows that $\mathcal{J}$ is G-injective. Now, let E be an object of $\mathcal{D}^+(\mathcal{E})$ and let

$$E \longrightarrow J$$

be an isomorphism where J is a complex of $\mathcal{I}$. We have

$$RG \circ \mathcal{D}(I)(E) \simeq G \circ I(J) \simeq F(J) \simeq RF(E)$$

as requested.

Assume now that

$$G' : \mathcal{LH}(\mathcal{E}) \longrightarrow \mathcal{LH}(\mathcal{F})$$

is an explicitly right derivable functor such that

$$RF \simeq RG' \circ \mathcal{D}(I).$$

It follows from this formula that

$$RG \circ \mathcal{D}(I) \simeq RG' \circ \mathcal{D}(I).$$

Since $\mathcal{D}(I)$ is an equivalence of categories, we see that

$$RG \simeq RG'.$$

Therefore G' is right equivalent to G.

The conclusion then follows from the definition of F^0. □

Proposition 1.3.15. *Let*

$$F : \mathcal{E} \longrightarrow \mathcal{F}$$

be an additive functor of quasi-abelian categories. Assume F is explicitly left derivable and consider its left derived functor

$$LF : \mathcal{D}^-(\mathcal{E}) \longrightarrow \mathcal{D}^-(\mathcal{F}).$$

Then, there exists an explicitly left derivable functor

$$G : \mathcal{LH}(\mathcal{E}) \longrightarrow \mathcal{LH}(\mathcal{F})$$

such that

$$LF = LG \circ \mathcal{D}(I)$$

and any such functor is left equivalent to the explicitly left derivable right exact functor

$$LH^0 \circ LF : \mathcal{LH}(\mathcal{E}) \to \mathcal{LH}(\mathcal{F}).$$

Moreover, the restriction of this functor to $\mathcal{E}$ is isomorphic to F if and only if F is regular and right exact.

Proof. Let $\mathcal{P}$ be a F-projective subcategory of $\mathcal{E}$. It follows from the construction of LF that it is right exact with respect to the left t-structures of $\mathcal{D}^-(\mathcal{E})$ and $\mathcal{D}^-(\mathcal{F})$. Therefore,

$$LH^0 \circ LF : \mathcal{LH}(\mathcal{E}) \to \mathcal{LH}(\mathcal{F})$$

is a right exact functor. Denote it by G and denote $\mathcal{Q}$ the essential image of $I_{|\mathcal{P}}$. Consider a short exact sequence of $\mathcal{LH}(\mathcal{E})$

$$0 \to A' \to A \to A'' \to 0$$

where A and A'' are objects of $\mathcal{Q}$. Since I is a fully faithful strictly exact functor, this sequence is isomorphic to the image by I of a strict exact sequence

$$0 \to E' \to E \to E'' \to 0$$

of $\mathcal{E}$ where E and E'' are objects of $\mathcal{P}$. It follows that E' is an object of $\mathcal{P}$ and consequently that A' is an object of $\mathcal{Q}$. Moreover, since the sequence

$$0 \to F(E') \to F(E) \to F(E'') \to 0$$

is strictly exact in $\mathcal{F}$, the sequence

$$0 \to G(A') \to G(A) \to G(A'') \to 0$$

is exact in $\mathcal{LH}(\mathcal{F})$. In order to show that $\mathcal{Q}$ is G-projective, it is thus sufficient to note that since any object A of $\mathcal{LH}(\mathcal{E})$ is a quotient of an object of the form $I(E)$ where E is an object of $\mathcal{E}$, it follows from the fact that $\mathcal{P}$ is F-projective that A is also a quotient of an object of $\mathcal{Q}$. The preceding discussion shows that G is explicitly left derivable. Consider the functor

$$LF : \mathcal{D}^-(\mathcal{E}) \to \mathcal{D}^-(\mathcal{F}).$$

Since $G \circ I(P) \simeq F(P)$ for any object P of $\mathcal{P}$, we get the requested isomorphism

$$LG \circ \mathcal{D}(I) \simeq LF.$$

Assume now that

$$G' : \mathcal{LH}(\mathcal{E}) \longrightarrow \mathcal{LH}(\mathcal{F})$$

is an explicitly left derivable functor such that

$$LF \simeq LG' \circ \mathcal{D}(I).$$

It follows from this formula that

$$LG \circ \mathcal{D}(I) \simeq LG' \circ \mathcal{D}(I).$$

Since $\mathcal{D}(I)$ is an equivalence of categories, we see that

$$LG \simeq LG'.$$

Therefore G' is left equivalent to G.

The last part of the result follows from Subsection 1.3.2 □

Proposition 1.3.16. *Let $\mathcal{E}$ and $\mathcal{F}$ be quasi-abelian categories and let*

$$F : \mathcal{E} \longrightarrow \mathcal{F} \qquad (\textit{resp. } E : \mathcal{F} \longrightarrow \mathcal{E})$$

be an explicitly left (resp. right) derivable functor. Assume that

$$\mathrm{Hom}_{\mathcal{F}}(F(X), Y) \simeq \mathrm{Hom}_{\mathcal{E}}(X, E(Y))$$

functorially in $X \in \mathcal{E}$, $Y \in \mathcal{F}$ (i.e. F is a left adjoint of E). Then,

$$\mathrm{Hom}_{D(\mathcal{F})}(LF(X), Y) \simeq \mathrm{Hom}_{D(\mathcal{E})}(X, RE(Y))$$

functorially in $X \in D^-(\mathcal{E})$, $Y \in D^+(\mathcal{F})$.

Proof. Let $\mathcal{P}$ be an F-projective subcategory of $\mathcal{E}$ and let $\mathcal{I}$ be an E-injective subcategory of $\mathcal{F}$. Using the canonical morphisms

$$\mathrm{id}_{D(\mathcal{F})} \longrightarrow \tau^{\geq n} \qquad (n \in \mathbb{Z})$$

and the properties of E-injective and F-projective subcategories, one checks easily that any morphism

$$u : LF(X) \to Y$$

of $D(\mathcal{F})$ may be embedded in a commutative diagram of the form

$$\begin{tikzcd} LF(X) \arrow[r, "u"] & Y \arrow[d, "\wr\alpha"] \\ F(P) \arrow[u, "\wr"] \arrow[r, "u'"'] & I \end{tikzcd}$$

where

(a) P (resp. I) comes from an object of $K^-(\mathcal{P})$ (resp. $K^+(\mathcal{I})$),

(b) the morphisms $u' : F(P) \to I$, $\alpha : Y \to I$ come from morphisms of $K(\mathcal{F})$,

(c) the isomorphism $F(P) \xrightarrow{\sim} LF(X)$ comes from a quasi-isomorphism

$$\beta : P \to X$$

of $K^-(\mathcal{E})$.

To such a diagram, we associate the unique morphism $v : X \to RE(Y)$ making the diagram

$$\begin{tikzcd} X \arrow[r, "v"] & RE(Y) \arrow[d, "\wr"] \\ P \arrow[u, "\beta"] \arrow[r, "v'"'] & E(I) \end{tikzcd}$$

commutative. Note that in this diagram $v' : P \to E(I)$ is obtained from u' by adjunction and that $RE(Y) \to E(I)$ is induced by α. We leave it to the reader to check that v depends only on u and that the process of passing from u to v defines a functorial morphism

$$\operatorname{Hom}_{D(\mathcal{F})}(LF(X), Y) \to \operatorname{Hom}_{D(\mathcal{E})}(X, RE(Y)).$$

Proceeding dually, we get a functorial morphism

$$\operatorname{Hom}_{D(\mathcal{E})}(X, RE(Y)) \to \operatorname{Hom}_{D(\mathcal{F})}(LF(X), Y)$$

and it is easy to check that this defines an inverse of the preceding one. The conclusion follows. □

Corollary 1.3.17. *In the situation of the preceding proposition,*

$$LH^0 \circ LF : \mathcal{LH}(\mathcal{E}) \to \mathcal{LH}(\mathcal{F})$$

is a left adjoint of

$$LH^0 \circ RE : \mathcal{LH}(\mathcal{F}) \to \mathcal{LH}(\mathcal{E}).$$

Proof. Since E is a right adjoint of F, E is strongly left exact. In particular,

$$RE(\mathcal{LH}(\mathcal{F})) \subset D^{\geq 0}(\mathcal{E})$$

for the left t-structure. The conclusion follows from the isomorphisms

$$\begin{aligned} \operatorname{Hom}_{\mathcal{LH}(\mathcal{F})}(LH^0 \circ LF(X), Y) &\simeq \operatorname{Hom}_{D^{\geq 0}(\mathcal{F})}(\tau^{\geq 0} \circ LF(X), Y) \\ &\simeq \operatorname{Hom}_{D(\mathcal{F})}(LF(X), Y) \end{aligned}$$

and

$$\begin{aligned}\operatorname{Hom}_{\mathcal{LH}(\mathcal{E})}(X, LH^0 \circ RE(Y)) &\simeq \operatorname{Hom}_{D^{\leq 0}(\mathcal{E})}(X, \tau^{\leq 0} \circ RE(Y)) \\ &\simeq \operatorname{Hom}_{D(\mathcal{E})}(X, RE(Y))\end{aligned}$$

holding for any $X \in \mathcal{LH}(\mathcal{E})$, $Y \in \mathcal{LH}(\mathcal{F})$. $\square$

1.3.4 Categories with enough projective or injective objects

Definition 1.3.18. An object I of $\mathcal{E}$ is *injective* (resp. *strongly injective*) if the functor

$$\operatorname{Hom}_{\mathcal{E}}(\cdot, I) : \mathcal{E}^{\mathrm{op}} \longrightarrow \mathcal{A}b$$

is exact (resp. strongly exact). Equivalently, I is injective (resp. strongly injective) if for any strict (resp. arbitrary) monomorphism

$$u : E \longrightarrow F$$

the associated map

$$\operatorname{Hom}(F, I) \longrightarrow \operatorname{Hom}(E, I)$$

is surjective.

Dually, an object P of $\mathcal{E}$ is *projective* (resp. *strongly projective*) if the functor

$$\operatorname{Hom}_{\mathcal{E}}(P, \cdot) : \mathcal{E} \longrightarrow \mathcal{A}b$$

is exact (resp. strongly exact). Equivalently, P is projective (resp. strongly projective) if for any strict (resp. arbitrary) epimorphism

$$u : E \longrightarrow F$$

the associated map

$$\operatorname{Hom}(P, E) \longrightarrow \operatorname{Hom}(P, F)$$

is surjective.

Remark 1.3.19. What we call a strongly projective object was simply called a projective object by some authors. We have chosen to stick to our definition for coherence with our notions of exact and strongly exact functor and also because projective objects are more frequent and more useful than strongly projective ones.

Definition 1.3.20. A quasi-abelian category $\mathcal{E}$ *has enough projective objects* if for any object E of $\mathcal{E}$ there is a strict epimorphism

$$P \longrightarrow E$$

where P is a projective object of $\mathcal{E}$.

Dually, a quasi-abelian category $\mathcal{E}$ *has enough injective objects* if for any object E of $\mathcal{E}$ there is a strict monomorphism

$$E \longrightarrow I$$

where I is an injective object of $\mathcal{E}$.

Remark 1.3.21. Let

$$F : \mathcal{E} \longrightarrow \mathcal{F}$$

be an additive functor.

Assume $\mathcal{E}$ has enough injective objects. Then, the full subcategory $\mathcal{I}$ of $\mathcal{E}$ formed by injective objects is an F-injective subcategory. In particular, F is explicitely right derivable.

Dually, assume $\mathcal{E}$ has enough projective objects. Then, the full subcategory $\mathcal{P}$ of $\mathcal{E}$ formed by projective objects is an F-projective subcategory. In particular, F is explicitely left derivable.

Proposition 1.3.22. *Let $\mathcal{P}$ (resp. $\mathcal{I}$) be a full additive subcategory of $\mathcal{E}$. Assume that:*

a) *The objects of $\mathcal{P}$ (resp. $\mathcal{I}$) are projective (resp. injective) in $\mathcal{E}$.*

b) *For any object E of $\mathcal{E}$, there is an object P of $\mathcal{P}$ (resp. I of $\mathcal{I}$) and a strict epimorphism (resp. monomorphism)*

$$P \longrightarrow E \quad (\textit{resp. } E \longrightarrow I).$$

Then the canonical functor

$$\mathcal{K}^-(\mathcal{P}) \longrightarrow \mathcal{D}^-(\mathcal{E}) \quad (\textit{resp. } \mathcal{K}^+(\mathcal{I}) \longrightarrow \mathcal{D}^+(\mathcal{E}))$$

is an equivalence of categories.

Proof. We treat only the part corresponding to $\mathcal{P}$. The statement for $\mathcal{I}$ will follow by duality.

Thanks to Lemma 1.3.3 the proof may proceed as in the abelian case and it is sufficient to show that strictly exact objects of $\mathcal{K}^-(\mathcal{P})$ are isomorphic to 0. Let P be an object of $\mathcal{K}^-(\mathcal{P})$. Assume P is strictly exact. By definition, this means that the sequences

$$0 \longrightarrow \operatorname{Ker} d^P_k \longrightarrow P_k \longrightarrow \operatorname{Ker} d^P_{k-1} \longrightarrow 0$$

are strictly exact for any $k \in \mathbb{Z}$. If $\operatorname{Ker} d^P_{k-1}$ is projective in $\mathcal{E}$, the sequence splits and $\operatorname{Ker} d^P_k$ is also projective. Therefore, a decreasing induction shows that the complex P is split and the conclusion follows by Remark 1.2.2. $\square$

Proposition 1.3.23. *Using the assumptions and notations of the preceding proposition, a sequence*

$$E' \xrightarrow{e'} E \xrightarrow{e''} E''$$

is strictly exact (resp. coexact) in $\mathcal{E}$ if and only if the sequence of abelian groups

$$\mathrm{Hom}\,(P, E') \longrightarrow \mathrm{Hom}\,(P, E) \longrightarrow \mathrm{Hom}\,(P, E'')$$

$$\text{(resp. } \mathrm{Hom}\,(E'', I) \longrightarrow \mathrm{Hom}\,(E, I) \longrightarrow \mathrm{Hom}\,(E', I)\ \text{)}$$

is exact for any $P \in \mathcal{P}$ (resp. $I \in \mathcal{I}$).

Proof. We consider only the case of $\mathcal{P}$, the other one is obtained by duality. The condition is clearly necessary, let us prove that it is also sufficient.

We will first show that a sequence

$$0 \longrightarrow E' \xrightarrow{e'} E \xrightarrow{e''} E''$$

is strictly exact if the sequence

$$0 \longrightarrow \mathrm{Hom}\,(P, E') \longrightarrow \mathrm{Hom}\,(P, E) \longrightarrow \mathrm{Hom}\,(P, E'')$$

is exact for any $P \in \mathcal{P}$. Let $x : X \longrightarrow E$ be a morphism of $\mathcal{E}$ such that $e'' \circ x = 0$. It follows from the preceding proposition that we may find a strict exact sequence of the form

$$P_1 \xrightarrow{\delta} P_0 \xrightarrow{\epsilon} X \longrightarrow 0$$

where P_1 and P_0 are in $\mathcal{P}$. It follows from our hypothesis that the sequence

$$0 \longrightarrow \mathrm{Hom}\,(P_k, E') \longrightarrow \mathrm{Hom}\,(P_k, E) \longrightarrow \mathrm{Hom}\,(P_k, E'')$$

is exact for $k \in \{0, 1\}$. Therefore, there is a morphism $x' : L_0 \longrightarrow E'$ such that $e' \circ x' = x \circ \epsilon$. Since

$$e' \circ x' \circ \delta = x \circ \epsilon \circ \delta = 0,$$

it follows that $x' \circ \delta = 0$. Hence, there is a morphism $x'' : X \longrightarrow E'$ such that $x'' \circ \epsilon = x'$. Clearly,

$$e' \circ x'' \circ \epsilon = e' \circ x' = x \circ \epsilon$$

and we see that $e' \circ x'' = x$. Since x'' is clearly the only morphism satisfying this property, it follows that e' is a kernel of e'' and the sequence

$$0 \longrightarrow E' \xrightarrow{e'} E \xrightarrow{e''} E''$$

is strictly exact.

To conclude, it is sufficient to show that a morphism

$$f : E \longrightarrow F$$

of $\mathcal{E}$ is a strict epimorphism if the associated morphism

$$\mathrm{Hom}\,(P, E) \longrightarrow \mathrm{Hom}\,(P, F)$$

is surjective for any $P \in \mathcal{P}$. But this is obvious since a relation of the form

$$\epsilon = f \circ \epsilon'$$

where $\epsilon : P \longrightarrow F$ is a strict epimorphism implies that f is itself a strict epimorphism. □

Proposition 1.3.24.

(a) *An object P of $\mathcal{E}$ is projective if and only if $I(P)$ is projective in $\mathcal{LH}(\mathcal{E})$.*

(b) *The category $\mathcal{E}$ has enough projective objects if and only $\mathcal{LH}(\mathcal{E})$ has enough projective objects. Moreover, in such a case, any projective object of $\mathcal{LH}(\mathcal{E})$ is isomorphic to an object of the form $I(P)$ where P is projective in $\mathcal{E}$.*

Proof. (a) Assume P is a projective in $\mathcal{E}$. Consider an epimorphism $u : A \longrightarrow B$ in $\mathcal{LH}(\mathcal{E})$ and a morphism $f : I(P) \longrightarrow B$. We have to show that f factors through u. Since we may replace A, B by isomorphic objects, we may assume that A and B are respectively represented by the monomorphisms $E_1 \xrightarrow{\delta_E} E_0$ and $F_1 \xrightarrow{\delta_F} F_0$ and that u comes from the morphism of complexes

$$\begin{tikzcd} F_1 \arrow[r, "\delta_F"] & F_0 \\ E_1 \arrow[u, "u_1"] \arrow[r, "\delta_E"] & E_0 \arrow[u, "u_0"'] \end{tikzcd}$$

We may also assume that f is represented by the morphism of complexes

$$\begin{tikzcd} F_1 \arrow[r, "\delta_F"] & F_0 \\ 0 \arrow[u] \arrow[r] & P \arrow[u, "\varphi"'] \end{tikzcd}$$

Since u is an epimorphism,

$$E_0 \oplus F_1 \xrightarrow{(u_0\ \delta_F)} F_0$$

is a strict epimorphism. It follows from the fact that P is projective in $\mathcal{E}$ that there are two morphisms $\varphi' : P \longrightarrow E_0$, $\varphi'' : P \longrightarrow F_1$ such that $\varphi = u_0 \circ \varphi' + \delta_F \circ \varphi''$. Therefore the morphism of complexes

$$\begin{tikzcd} E_1 \arrow[r, "\delta_E"] & E_0 \\ 0 \arrow[u] \arrow[r] & P \arrow[u, "\varphi'"'] \end{tikzcd}$$

induces a morphism $f' : I(P) \longrightarrow A$ such that $u \circ f' = f$.

Since I is an exact functor, it follows from the adjunction formula

$$\mathrm{Hom}_{\mathcal{LH}(\mathcal{E})}(C(A), E) \simeq \mathrm{Hom}_{\mathcal{E}}(A, I(E))$$

that C transforms a projective object of $\mathcal{LH}(\mathcal{E})$ into a projective object of $\mathcal{E}$. Therefore an object P of $\mathcal{E}$ such that $I(P)$ is projective in $\mathcal{LH}(\mathcal{E})$ is projective in $\mathcal{E}$.

(b) Assume $\mathcal{E}$ has enough projective objects and let A be an object of $\mathcal{LH}(\mathcal{E})$. We know that there is an epimorphism

$$I(E) \rightarrow A$$

where E is an object of $\mathcal{E}$. Choose a strict epimorphism in $\mathcal{E}$

$$P \rightarrow E$$

where P is projective. We know that,

$$I(P) \rightarrow I(E)$$

is an epimorphism in $\mathcal{LH}(\mathcal{E})$ and that $I(P)$ is projective. Therefore, $\mathcal{LH}(\mathcal{E})$ has enough projective objects.

Assume now that $\mathcal{LH}(\mathcal{E})$ has enough projective objects and let E be an object of $\mathcal{E}$. There is an epimorphism

$$P \rightarrow I(E)$$

in $\mathcal{LH}(\mathcal{E})$ where P is a projective object in $\mathcal{LH}(\mathcal{E})$. Since C has a right adjoint, it is cokernel preserving and transforms epimorphisms in $\mathcal{LH}(\mathcal{E})$ into strict epimorphisms in $\mathcal{E}$. Therefore,

$$C(P) \rightarrow E$$

is a strict epimorphism. Since we have already remarked that $C(P)$ is a projective object of $\mathcal{E}$, it is clear that $\mathcal{E}$ has enough projective objects.

Since any projective object Q of $\mathcal{LH}(\mathcal{E})$ is a quotient of an object of the form $I(P)$ where P is a projective object of $\mathcal{E}$, it is a direct summand of such and object. It follows from Proposition 1.2.28 and part (a) that it is itself isomorphic to the image by I of a projective object of $\mathcal{E}$. $\square$

Remark 1.3.25. Assume $\mathcal{E}$ has enough projective objects. For any object E of $\mathcal{E}$ and any injective object I of $\mathcal{E}$, we have

$$\mathrm{Hom}_{\mathcal{E}}(E, I) \simeq \mathrm{RHom}_{\mathcal{E}}(E, I) \simeq \mathrm{RHom}_{\mathcal{LH}(\mathcal{E})}(I(E), I(I)).$$

Therefore, $\mathrm{Ext}^j_{\mathcal{LH}(\mathcal{E})}(I(E), I(I))$ vanish for $j > 0$. Nevertheless, $I(I)$ is not in general injective in $\mathcal{LH}(\mathcal{E})$.

Proposition 1.3.26.

(a) *An object J of $\mathcal{E}$ is strongly injective if and only if $I(J)$ is injective in $\mathcal{LH}(\mathcal{E})$.*

(b) *Assume that for any object E of $\mathcal{E}$ there is a strict monomorphism*

$$E \to J$$

where J is a strongly injective object of $\mathcal{E}$. Then, $\mathcal{E}$ is abelian.

Proof. (a) Let J be an object of $\mathcal{E}$ and assume $I(J)$ is injective in $\mathcal{LH}(\mathcal{E})$. Let

$$E \to F$$

be a monomorphism in $\mathcal{E}$. We know that

$$I(E) \to I(F)$$

is a monomorphism in $\mathcal{LH}(\mathcal{E})$. Therefore,

$$\mathrm{Hom}_{\mathcal{LH}(\mathcal{E})}(I(F), I(J)) \to \mathrm{Hom}_{\mathcal{LH}(\mathcal{E})}(I(E), I(J))$$

is surjective. Since the functor

$$I : \mathcal{E} \to \mathcal{LH}(\mathcal{E})$$

is fully faithful, it follows that

$$\mathrm{Hom}_{\mathcal{E}}(F, J) \to \mathrm{Hom}_{\mathcal{E}}(E, J)$$

is surjective. This shows that J is strongly injective in $\mathcal{E}$.

Assume now that J is a strongly injective object of $\mathcal{E}$. Up to isomorphism, a monomorphism u of $\mathcal{LH}(\mathcal{E})$ is represented by a cartesian square

$$\begin{tikzcd} E_0 \arrow[r, "u_0"] & F_0 \\ E_1 \arrow[u, "\delta_E"] \arrow[r, "u_1"] & F_1 \arrow[u, "\delta_F"'] \end{tikzcd}$$

whose associated sequence

$$0 \longrightarrow E_1 \xrightarrow{\binom{\delta_E}{u_1}} E_0 \oplus F_1 \xrightarrow{(\, u_0 \ -\delta_F \,)} F_0$$

is thus strictly exact. Denote α and β the second and third morphism of the preceding sequence and denote γ the canonical morphism

$$E_0 \oplus F_1 \longrightarrow \mathrm{Coim}\beta.$$

In $\mathcal{LH}(\mathcal{E})$, a morphism f from

$$E_1 \xrightarrow{\delta_E} E_0$$

to $I(J)$ is given by a morphism

$$f_0 : E_0 \longrightarrow J$$

such that $f_0 \circ \delta_E = 0$. Denote g the morphism

$$E_0 \oplus F_1 \xrightarrow{(\, f_0 \ 0 \,)} J.$$

Since $g \circ \alpha = 0$, there is a morphism $g' : \mathrm{Coim}\beta \longrightarrow J$ such that

$$g = g' \circ \gamma.$$

Since the canonical morphism

$$\mathrm{Coim}\beta \longrightarrow F_0$$

is a monomorphism in $\mathcal{E}$ and J is strongly injective in $\mathcal{E}$, we can extend g' into a morphism $g'' : F_0 \longrightarrow J$. Clearly, this morphism induces a morphism h from

$$F_1 \xrightarrow{\delta_F} F_0$$

to $I(J)$ in $\mathcal{LH}(\mathcal{E})$ such that $f = h \circ u$. Hence $I(J)$ is injective in $\mathcal{LH}(\mathcal{E})$.

(b) We know that for any complex E of $\mathcal{E}$ there is a complex J of $\mathcal{E}$ and a quasi-isomorphism

$$u : E \longrightarrow J$$

such that, for any $k \in \mathbb{Z}$, J^k is a strong injective object of $\mathcal{E}$ and

$$u^k : E^k \longrightarrow J^k$$

is a strict monomorphism. Let

$$E_1 \xrightarrow{\delta} E_0$$

be a bimorphism of $\mathcal{E}$ and denote by E the associated object of $\mathcal{LH}(\mathcal{E})$. From what precedes, we may find a complex J and a quasi-isomorphism

$$u : E \longrightarrow J$$

such that

$$u_1 : E_1 \longrightarrow J_1, \quad u_0 : E_0 \longrightarrow J_0$$

are strict monomorphisms and J^k are strong injective objects of $\mathcal{E}$. Hence, the complex

$$J_1 \xrightarrow{\delta'} \mathrm{Ker} d_J^0$$

is isomorphic to E and δ' is a bimorphism. Since J_1 is a strong injective object, δ' has an inverse in $\mathcal{E}$. It follows that E is quasi-isomorphic to 0. Therefore δ is an isomorphism of $\mathcal{E}$. The conclusion follows easily. □

1.4 Limits in quasi-abelian categories

1.4.1 Product and direct sums

In this subsection, we study products in quasi-abelian categories. Our results show mainly that a quasi-abelian category $\mathcal{E}$ has exact (resp. strongly exact) products if and only if $\mathcal{LH}(\mathcal{E})$ (resp. $\mathcal{RH}(\mathcal{E})$) has exact products and the canonical functor

$$\mathcal{E} \longrightarrow \mathcal{LH}(\mathcal{E}) \quad (\text{resp. } \mathcal{E} \longrightarrow \mathcal{RH}(\mathcal{E}))$$

is product preserving. We also give criteria for these conditions to be satisfied. We leave it to the reader to state the dual results for direct sums.

Lemma 1.4.1. *Let $\mathcal{A}$ be an additive category and let I be a small set. Then, $\mathcal{A}^I$ is an additive category and the canonical functor*

$$\mathcal{K}(\mathcal{A}^I) \to \mathcal{K}(\mathcal{A})^I$$

is an equivalence of triangulated categories.

Proof. For any $C \in \mathcal{C}(\mathcal{A}^I)$ and any $i \in I$, denote C_i the complex of $\mathcal{A}$ defined by setting

$$C_i^n = (C^n)_i$$

$$d_{C_i}^n = (d_C^n)_i.$$

This gives us a canonical functor

$$\begin{aligned} C(\mathcal{A}^I) &\to C(\mathcal{A})^I \\ C &\mapsto (C_i)_{i\in I} \end{aligned}$$

which is trivially an isomorphism of categories. Since two morphisms

$$f : C \to D \ , \quad g : C \to D$$

are homotopic in $\mathcal{C}(\mathcal{A}^I)$ if and only if

$$f_i : C_i \to D_i \ , \quad g_i : C_i \to D_i$$

are homotopic in $\mathcal{C}(\mathcal{A})$ for every $i \in I$, it is clear that we have a canonical isomorphism of categories

$$\mathcal{K}(\mathcal{A}^I) \xrightarrow{\sim} \mathcal{K}(\mathcal{A})^I.$$

In $\mathcal{A}^I$, we have

$$(A \oplus B)_i = A_i \oplus B_i.$$

Hence, the preceding functor exchanges the distinguished triangles of $\mathcal{K}(\mathcal{A}^I)$ with the distinguished triangles of $\mathcal{K}(\mathcal{A})^I$. □

Lemma 1.4.2. *Let $\mathcal{A}$ be an additive category with products. Then, both $\mathcal{C}(\mathcal{A})$ and $\mathcal{K}(\mathcal{A})$ have products.*

Proof. The functor

$$\prod_{i\in I} : \mathcal{A}^I \to \mathcal{A}$$

being additive gives rise to a functor

$$\mathcal{C}(\prod_{i\in I}) : \mathcal{C}(\mathcal{A}^I) \to \mathcal{C}(\mathcal{A})$$

and to a functor

$$\mathcal{K}(\prod_{i\in I}) : \mathcal{K}(\mathcal{A}^I) \to \mathcal{K}(\mathcal{A}).$$

By composition with the canonical equivalences

$$\mathcal{C}(\mathcal{A})^I \to \mathcal{C}(\mathcal{A}^I)$$

and

$$\mathcal{K}(\mathcal{A})^I \to \mathcal{K}(\mathcal{A}^I),$$

this gives us functors

$$\prod_{i\in I} : \mathcal{C}(\mathcal{A})^I \to \mathcal{C}(\mathcal{A})$$

$$\prod_{i\in I} : \mathcal{K}(\mathcal{A})^I \to \mathcal{K}(\mathcal{A})$$

which are easily checked to be product functors for the corresponding categories. □

Proposition 1.4.3. *Let $\mathcal{E}$ be a quasi-abelian category and let I be a small set. Then, $\mathcal{E}^I$ is quasi-abelian and the canonical functor*

$$\mathcal{D}(\mathcal{E}^I) \to \mathcal{D}(\mathcal{E})^I$$

is an equivalence of triangulated categories which is compatible with the left and right t-structures. In particular, we have canonical equivalences

$$\mathcal{LH}(\mathcal{E}^I) \approx \mathcal{LH}(\mathcal{E})^I, \qquad \mathcal{RH}(\mathcal{E}^I) \approx \mathcal{RH}(\mathcal{E})^I.$$

Proof. For any morphism $f : E \to F$ of $\mathcal{E}^I$, we have

$$(\mathrm{Ker} f)_i = \mathrm{Ker} f_i \quad \text{and} \quad (\mathrm{Coker} f)_i = \mathrm{Coker} f_i.$$

Therefore, an object E of $\mathcal{K}(\mathcal{E}^I)$ is strictly exact in degree n if and only if the complex C_i is strictly exact in degree n for any $i \in I$. The conclusion follows easily from this fact. □

Definition 1.4.4. A quasi-abelian category $\mathcal{E}$ has *exact (resp. strongly exact) products* if it is complete and if the functor

$$\prod_{i\in I} : \mathcal{E}^I \to \mathcal{E}$$

is exact (resp. strongly exact) for any small set I.

Proposition 1.4.5. *Let $\mathcal{E}$ be a complete quasi-abelian category. Assume $\mathcal{E}$ has enough projective objects. Then, $\mathcal{E}$ has exact products.*

Proof. Let

$$u_i : E_i \to F_i \qquad (i \in I)$$

be a small family of strict epimorphisms of $\mathcal{E}$. We have to prove that

$$\prod_{i\in I} E_i \xrightarrow{\prod_{i\in I} u_i} \prod_{i\in I} F_i$$

is a strict epimorphism. Let

$$v : P \longrightarrow \prod_{i \in I} F_i$$

be a strict epimorphism where P is a projective object of $\mathcal{E}$. For any $i \in I$, it follows from our assumptions that there is $w_i : P \longrightarrow E_i$ such that

$$p_i \circ v = u_i \circ w_i.$$

Let $w : P \longrightarrow \prod_{i \in I} E_i$ be the unique morphism such that

$$p_i \circ w = w_i.$$

Clearly,

$$\left(\prod_{i \in I} u_i \right) \circ w = v$$

and the conclusion follows from the fact that v is a strict epimorphism. $\square$

Proposition 1.4.6. *Assume $\mathcal{E}$ is a quasi-abelian category with exact products. Then, the category $\mathcal{D}(\mathcal{E})$ has products. Moreover, for any small set I,*

$$\prod_{i \in I} : \mathcal{D}(\mathcal{E})^I \to \mathcal{D}(\mathcal{E})$$

is a triangulated functor which is exact for the left t-structures. It is exact for the right t-structures, if and only if products are strongly exact in $\mathcal{E}$.

Proof. Since the functor

$$\prod_{i \in I} : \mathcal{E}^I \to \mathcal{E}$$

is exact, it gives rise to a functor

$$\mathcal{D}(\prod_{i \in I}) : \mathcal{D}(\mathcal{E}^I) \to \mathcal{D}(\mathcal{E}).$$

By composition with the canonical equivalence

$$\mathcal{D}(\mathcal{E})^I \approx \mathcal{D}(\mathcal{E}^I)$$

this gives us a functor

$$P_{i \in I} : \mathcal{D}(\mathcal{E})^I \to \mathcal{D}(\mathcal{E}).$$

We will show that this is a product functor for $\mathcal{D}(\mathcal{E})$.

Let E be an object of $\mathcal{D}(\mathcal{E})$ and let F be an object of $\mathcal{D}(\mathcal{E})^I$. We have to prove that the canonical morphism

$$\operatorname{Hom}_{\mathcal{D}(\mathcal{E})}(E, P_{i\in I}F_i) \longrightarrow \prod_{i\in I} \operatorname{Hom}_{\mathcal{D}(\mathcal{E})}(E, F_i) \tag{*}$$

is bijective.

Let

$$E \xrightarrow{f_i} F_i$$

be a family of morphisms of $\mathcal{D}(\mathcal{E})$. We may assume that there is a strict quasi-isomorphism

$$s : F \longrightarrow G$$

of $\mathcal{K}(\mathcal{E}^I)$ and a family of morphisms

$$g_i : E \longrightarrow G_i \qquad (i \in I)$$

of $\mathcal{K}(\mathcal{E})$ such that the morphism f_i is represented by the diagram

$$\begin{tikzcd} & G_i & \\ E \arrow[ur, "g_i"] & & F_i \arrow[ul, "s_i"'] \end{tikzcd}$$

for each $i \in I$. Denote

$$g : E \longrightarrow \prod_{i\in I} G_i$$

the morphism of $\mathcal{K}(\mathcal{E})$ associated to the family $(g_i)_{i\in I}$. Since $P_{i\in I}$ is a functor, $\prod_{i\in I} s_i$ is a strict quasi-isomorphism of $\mathcal{K}(\mathcal{E})$. Hence, we may define

$$f : E \longrightarrow P_{i\in I}F_i$$

as the morphism of $\mathcal{D}(\mathcal{E})$ represented by the diagram

$$\begin{tikzcd} & \prod_{i\in I} G_i & \\ E \arrow[ur, "g"] & & \prod_{i\in I} F_i \arrow[ul, "\prod_{i\in I} s_i"'] \end{tikzcd}$$

Since the diagram

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[dr, "g_i"'] & \prod_{i\in I} G_i \arrow[d, "p_i"] & \prod_{i\in I} F_i \arrow[l, "\prod_{i\in I} s_i"'] \arrow[d, "p_i"] \\ & G_i & F_i \arrow[l, "s_i"] \end{tikzcd}$$

commutes in $\mathcal{K}(\mathcal{E})$, we see that the diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[dr, "f_i"'] & \prod_{i\in I}(F_i) \arrow[d, "p_i"] \\ & F_i \end{tikzcd}$$

is commutative in $\mathcal{D}(\mathcal{E})$. This shows that (*) is surjective.

To show that (*) is injective, we have to prove that if

$$f : E \longrightarrow \prod_{i\in I} F$$

is a morphism of $\mathcal{D}(\mathcal{E})$ such that the diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[dr, "0"'] & \prod_{i\in I} F_i \arrow[d, "p_i"] \\ & F_i \end{tikzcd}$$

commutes in $\mathcal{D}(\mathcal{E})$ for every $i \in I$, then $f = 0$. Let

$$\begin{tikzcd} & G & \\ E \arrow[ur, "f"] & & \prod_{i\in I} F_i \arrow[ul, "s"'] \end{tikzcd}$$

be a diagram of $\mathcal{K}(\mathcal{E})$ representing f.

Recall that a diagram

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "z"] & & Y \arrow[ul, "s"'] \end{tikzcd}$$

of $\mathcal{K}(\mathcal{E})$ where s is a strict quasi-isomorphism, represents the morphism

$$X \xrightarrow{0} Y$$

of $\mathcal{D}(\mathcal{E})$ if and only if there is a commutative diagram of $\mathcal{K}(\mathcal{E})$ of the form

$$\begin{tikzcd} & Z \arrow[dd] & \\ X \arrow[ur, "z"] \arrow[dr, "0"'] & & Y \arrow[ul, "s"'] \arrow[dl, "t"] \\ & Z' & \end{tikzcd}$$

where t is a strict quasi-isomorphism.

For any $i \in I$, we have $p_i \circ f = 0$. Therefore, there is in $\mathcal{K}(\mathcal{E})$ a commutative diagram of the form

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[dr, "0"'] & G \arrow[d, "z_i"] & \prod_{i \in I} F_i \arrow[l, "s"'] \arrow[d, "p_i"] \\ & Z_i & F_i \arrow[l, "t_i"'] \end{tikzcd}$$

where t_i is a strict quasi-isomorphism. Denoting

$$z : G \longrightarrow \prod_{i \in I} Z_i$$

the morphism of $\mathcal{K}(\mathcal{E})$ associated to $(z_i)_{i \in I}$, we get, in $\mathcal{K}(\mathcal{E})$, the commutative diagram

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[dr, "0"'] & G \arrow[d, "z"] & \prod_{i \in I} F_i \arrow[l, "s"'] \arrow[d, "\wr"] \\ & \prod_{i \in I} Z_i & \prod_{i \in I} F_i \arrow[l, "\prod_{i \in I} t_i"'] \end{tikzcd}$$

Since $P_{i \in I}$ is a functor, $\prod_{i \in I} t_i$ is a strict quasi-isomorphism. Therefore, $f = 0$ in $\mathcal{D}(\mathcal{E})$.

Since product functors are always strongly left exact, the last part of the proposition is clear. $\square$

Corollary 1.4.7. *Assume $\mathcal{E}$ is a quasi-abelian category with exact products. Then,*

(a) The abelian category $\mathcal{LH}(\mathcal{E})$ (resp. $\mathcal{RH}(\mathcal{E})$) is complete and the canonical functor

$$\mathcal{E} \longrightarrow \mathcal{LH}(\mathcal{E}) \quad (\text{resp. } \mathcal{E} \longrightarrow \mathcal{RH}(\mathcal{E}))$$

is product preserving.

(b) Products are exact in $\mathcal{LH}(\mathcal{E})$.

(c) Products are exact in $\mathcal{RH}(\mathcal{E})$ if and only if they are strongly exact in $\mathcal{E}$.

Proof. Let $(A_i)_{i \in I}$ be a family of objects of $\mathcal{LH}(\mathcal{E})$. We know that A_i may be represented by a monomorphism

$$E_i \longrightarrow F_i$$

of $\mathcal{E}$. Hence, it is clear that the object

$$\prod_{i\in I} A_i$$

of $\mathcal{D}(\mathcal{E})$ is isomorphic to the complex

$$\prod_{i\in I} E_i \longrightarrow \prod_{i\in I} F_i.$$

Products being strongly left exact functors, this complex is in $\mathcal{LH}(\mathcal{E})$. Since $\mathcal{LH}(\mathcal{E})$ is a full subcategory of $\mathcal{D}(\mathcal{E})$, it follows that $\prod_{i\in I} A_i$ is the product of the family $(A_i)_{i\in I}$ in $\mathcal{LH}(\mathcal{E})$. Using the fact that

$$\prod_{i\in I} : \mathcal{D}(\mathcal{E})^I \longrightarrow \mathcal{D}(\mathcal{E})$$

is triangulated, we check easily that products are exact in $\mathcal{LH}(\mathcal{E})$. This proves (b).

Let $(A_i)_{i\in I}$ be a family of objects of $\mathcal{RH}(\mathcal{E})$. We know that A_i may be represented by an epimorphism

$$E_i \longrightarrow F_i$$

of $\mathcal{E}$. Hence, it is clear that the object $\prod_{i\in I} A_i$ of $\mathcal{D}(\mathcal{E})$ is isomorphic to the complex

$$\prod_{i\in I} E_i \longrightarrow \prod_{i\in I} F_i.$$

This complex has components in degree 0 and 1. Hence, it is in $\mathcal{D}^{\geq 0}(\mathcal{E})$ for the right t-structure of $\mathcal{D}(\mathcal{E})$. It follows that

$$\prod_{i\in I} \mathrm{Hom}_{\mathcal{RH}(\mathcal{E})}(X, A_i) \simeq \mathrm{Hom}_{\mathcal{D}(\mathcal{E})}(X, \prod_{i\in I} A_i) \simeq \mathrm{Hom}_{\mathcal{RH}(\mathcal{E})}(X, RH^0(\prod_{i\in I} A_i)).$$

Hence,

$$RH^0(\prod_{i\in I} A_i)$$

is a product of the family $(A_i)_{i\in I}$ in $\mathcal{RH}(\mathcal{E})$. If products are strongly exact in $\mathcal{E}$,

$$RH^0(\prod_{i\in I} A_i) \simeq \prod_{i\in I} A_i$$

and the exactness of products in $\mathcal{RH}(\mathcal{E})$ follows as in the case of $\mathcal{LH}(\mathcal{E})$. Conversely, if products are exact in $\mathcal{RH}(\mathcal{E})$, a family of morphisms $(u_i)_{i\in I}$ of $\mathcal{E}$ gives rise to the exact sequences

$$I(E_i) \xrightarrow{I(u_i)} I(F_i) \longrightarrow I(\mathrm{Coker} u_i) \longrightarrow 0$$

in $\mathcal{RH}(\mathcal{E})$ and since

$$\prod_{i\in I} I(E_i) \to \prod_{i\in I} I(F_i) \to \prod_{i\in I} I(\mathrm{Coker} u_i) \to 0$$

is exact in $\mathcal{RH}(\mathcal{E})$, one sees that products are strongly exact in $\mathcal{E}$.

To conclude, it remains to note that thanks to the preceding constructions of products in $\mathcal{LH}(\mathcal{E})$ and $\mathcal{RH}(\mathcal{E})$, the last part of (a) is obvious. □

Proposition 1.4.8. *Let $\mathcal{E}$ be a quasi-abelian category and assume that $\mathcal{LH}(\mathcal{E})$ (resp. $\mathcal{RH}(\mathcal{E})$) is complete. Then, $\mathcal{E}$ is complete.*

Proof. Let us prove that $\mathcal{E}$ has products if so has $\mathcal{LH}(\mathcal{E})$. Let $(E_j)_{j\in J}$ be a small family of objects of $\mathcal{E}$. Let

$$p_j : \prod_{j\in J} I(E_j) \to I(E_j)$$

denote the canonical projections of the product to its factors. Let

$$q_j : C(\prod_{j\in J} I(E_j)) \to E_j$$

denote the morphism obtained by composing $C(p_j)$ with the canonical isomorphism

$$C \circ I(E_j) \xrightarrow{\sim} E_j.$$

We will prove that

$$C(\prod_{j\in J} I(E_j))$$

together with the projections q_j form a product of the family $(E_j)_{j\in J}$.

First, let

$$x_j : X \to E_j \qquad (j \in J)$$

be a family of morphisms of $\mathcal{E}$. Denote

$$x' : I(X) \to \prod_{j\in J} I(E_j)$$

the unique morphism of $\mathcal{LH}(\mathcal{E})$ such that

$$p_j \circ x' = I(x_j).$$

Let

$$x : X \to C(\prod_{j\in J} I(E_j))$$

be the morphism obtained by composing $C(x')$ with the canonical isomorphism

$$X \xrightarrow{\sim} C \circ I(X).$$

Clearly,

$$q_j \circ x = x_j$$

for $j \in J$.

Next, let

$$x : X \longrightarrow C(\prod_{j \in J} I(E_j))$$

be a morphism such that

$$q_j \circ x = 0 \qquad (j \in J).$$

In $\mathcal{LH}(\mathcal{E})$, let us form the cartesian square

$$\begin{tikzcd}
\prod_{j\in J} I(E_j) \arrow[r, "y"] & I \circ C(\prod_{j\in J} I(E_j)) \\
Y \arrow[u, "u"] \arrow[r, "v"'] & I(X) \arrow[u, "I(x)"']
\end{tikzcd}$$

where the first horizontal arrow is the canonical epimorphism. Since

$$I(q_j) \circ y = p_j$$

it is clear that

$$p_j \circ u = I(q_j) \circ y \circ u = I(q_j \circ x) \circ v = 0$$

and we deduce that $u = 0$. Therefore,

$$I(x) \circ v = y \circ u = 0.$$

Since y is an epimorphism, so is v and we get $I(x) = 0$. Hence, $x = 0$.

To conclude, let us prove that $\mathcal{E}$ has products if so has $\mathcal{RH}(\mathcal{E})$. By duality it is equivalent to show that $\mathcal{E}$ has direct sums when $\mathcal{LH}(\mathcal{E})$ has direct sums. From the adjunction formula

$$\mathrm{Hom}_{\mathcal{E}}(C(A), E) \simeq \mathrm{Hom}_{\mathcal{LH}(\mathcal{E})}(A, I(E))$$

it follows that, for any small family $(E_i)_{i \in I}$ and any object X of $\mathcal{E}$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{E}}(C(\bigoplus_{i\in I} I(E_i)), X) &\simeq \mathrm{Hom}_{\mathcal{LH}(\mathcal{E})}(\bigoplus_{i\in I} I(E_i), I(X)) \\
&\simeq \prod_{i\in I} \mathrm{Hom}_{\mathcal{LH}(\mathcal{E})}(I(E_i), I(X)) \\
&\simeq \prod_{i\in I} \mathrm{Hom}_{\mathcal{E}}(E_i, X).
\end{aligned}$$

Hence,

$$C(\bigoplus_{i\in I} I(E_i))$$

is a direct sum of the family $(E_i)_{i\in I}$ in $\mathcal{E}$. □

1.4.2 Projective and inductive systems

In this subsection, we study categories of projective systems of a quasi-abelian category. We leave it to the reader to state the dual results for inductive systems.

Let $\mathcal{E}$ be a quasi-abelian category and let $\mathcal{I}$ be a small category. Recall that the category of projective systems of $\mathcal{E}$ indexed by $\mathcal{I}$ is the category

$$\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$$

of contravariant functors from $\mathcal{I}$ to $\mathcal{E}$.

Proposition 1.4.9. *Let $\mathcal{E}$ be a quasi-abelian category and let $\mathcal{I}$ be a small category. Then, the category of projective systems of $\mathcal{E}$ indexed by $\mathcal{I}$ is quasi-abelian.*

Proof. Since for any morphism

$$u : E \longrightarrow F$$

of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$, we have

$$(\mathrm{Ker}u)(i) \simeq \mathrm{Ker}u(i),$$

$$(\mathrm{Coker}u)(i) \simeq \mathrm{Coker}u(i).$$

the conclusion follows easily. □

Definition 1.4.10. Let F be an object of $\mathcal{E}$ and let i be an object of $\mathcal{I}$.

Assuming $\mathcal{E}$ is complete, we denote F^i the object of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ defined by setting

$$F^i(i') = F^{\mathrm{Hom}_{\mathcal{I}}(i,i')} = \prod_{\alpha\in \mathrm{Hom}_{\mathcal{I}}(i,i')} F.$$

A projective system isomorphic to a system of the form F^i will be said to be of *elementary type*.

Similarly, assuming $\mathcal{E}$ is cocomplete, we denote F_i the object of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ defined by setting

$$F_i(i') = F^{(\mathrm{Hom}_{\mathcal{I}}(i',i))} = \bigoplus_{\alpha\in \mathrm{Hom}_{\mathcal{I}}(i',i)} F.$$

A projective system isomorphic to a system of the form F_i will be said to be of *coelementary type*.

Remark 1.4.11. With the notation of the preceding proposition, F^i is exchanged with F_i if one exchanges $\mathcal{E}$ with $\mathcal{E}^{\mathrm{op}}$ and $\mathcal{I}$ with $\mathcal{I}^{\mathrm{op}}$.

Proposition 1.4.12. *If $\mathcal{E}$ is complete, then*

$$\mathrm{Hom}_{\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}}(E, F^i) \simeq \mathrm{Hom}_{\mathcal{E}}(E(i), F)$$

and there is a strict monomorphism

$$E \longrightarrow \prod_{i \in \mathcal{I}} E(i)^i$$

for any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$.

Similarly, if $\mathcal{E}$ is cocomplete, then

$$\mathrm{Hom}_{\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}}(F_i, E) \simeq \mathrm{Hom}_{\mathcal{E}}(F, E(i))$$

and there is a strict epimorphism

$$\bigoplus_{i \in \mathcal{I}} E(i)_i \longrightarrow E$$

for any object E of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$.

Proof. Thanks to the preceding remark, it is sufficient to prove the first statement.

For any $i' \in \mathcal{I}$, we have

$$\mathrm{Hom}_{\mathcal{E}}(E(i'), F^i(i')) \simeq \mathrm{Hom}_{\mathcal{S}et}(\mathrm{Hom}\,(i, i'), \mathrm{Hom}\,(E(i'), F)).$$

These isomorphisms give us the isomorphism

$$\mathrm{Hom}_{\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}}(E, F^i) \simeq \mathrm{Hom}_{\mathcal{S}et^{\mathcal{I}}}(h_i, h^F \circ E).$$

Using standard results on representable functors, we get

$$\mathrm{Hom}_{\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}}(E, F^i) \simeq h^F \circ E(i) \simeq \mathrm{Hom}\,(E(i), F).$$

Now let us prove the second part of the result. For any $i \in I$, the identity morphism

$$E(i) \longrightarrow E(i)$$

induces, by an isomorphism of the preceding kind, a morphism

$$E \longrightarrow E(i)^i.$$

Together, these morphisms give us a canonical morphism

$$u : E \longrightarrow \prod_{i \in I} E(i)^i.$$

Choose $i' \in I$. To conclude, we have to prove that $u(i')$ is a strict monomorphism. Note that

$$(\prod_{i \in \mathcal{I}} E(i)^i)(i') \simeq \prod_{i \in \mathcal{I}} E(i)^i(i') \simeq \prod_{i \in \mathcal{I}} \prod_{\alpha \in \mathrm{Hom}_{\mathcal{I}}(i,i')} E(i).$$

Composing $u(i')$ with this isomorphism, we get a morphism

$$v(i') : E(i') \longrightarrow \prod_{i \in \mathcal{I}} \prod_{\alpha \in \mathrm{Hom}_{\mathcal{I}}(i,i')} E(i)$$

such that

$$p_\alpha \circ p_i \circ v(i') = E(\alpha)$$

for any $\alpha : i \longrightarrow i'$ in $\mathcal{I}$. For $\alpha = \mathrm{id}_{i'}$ this shows that $v(i')$ is a strict monomorphism and the conclusion follows. $\square$

Remark 1.4.13. Taking E to be a constant functor in the preceding proposition shows that

$$\varprojlim_{i' \in \mathcal{I}} F^i(i') \simeq F$$

if $\mathcal{E}$ is complete.

Corollary 1.4.14. *Let $\mathcal{E}$ be a complete (resp. cocomplete) quasi-abelian category.*

(a) Assume F is an injective (resp. a projective) object of $\mathcal{E}$. Then, for any $i \in \mathcal{I}$, F^i (resp. F_i) is injective (resp. projective) in $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$.

(b) Assume $\mathcal{E}$ has enough injective (resp. projective) objects. Then $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ has enough injective (resp. projective) objects

Proposition 1.4.15. *Let $\mathcal{E}$ be a quasi-abelian category with exact direct sums and let $\mathcal{I}$ be a small category. Then, there is a canonical equivalence*

$$\mathcal{LH}(\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}) \approx \mathcal{LH}(\mathcal{E})^{\mathcal{I}^{\mathrm{op}}}.$$

Proof. The canonical inclusion functor

$$I : \mathcal{E} \longrightarrow \mathcal{LH}(\mathcal{E})$$

gives rise to a fully faithful functor

$$I : \mathcal{E}^{\mathcal{I}^{\mathrm{op}}} \longrightarrow \mathcal{LH}(\mathcal{E})^{\mathcal{I}^{\mathrm{op}}}.$$

Let E be an object of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ and let

$$A \longrightarrow I(E)$$

be a monomorphism of $\mathcal{LH}(\mathcal{E})^{\mathcal{I}^{\mathrm{op}}}$. Since

$$A(i) \longrightarrow I(E(i))$$

is a monomorphism of $\mathcal{LH}(\mathcal{E})$ for any $i \in \mathcal{I}$, we can find for any $i \in \mathcal{I}$ an object $E'(i)$ of $\mathcal{E}$ and an isomorphism

$$I(E'(i)) \simeq A(i).$$

Using these isomorphisms, we may turn E' into an object of $\mathcal{E}^{\mathcal{I}^{\mathrm{op}}}$ such that

$$I(E') \simeq A.$$

Therefore, $I(\mathcal{E}^{\mathcal{I}^{\mathrm{op}}})$ is a full subcategory of $\mathcal{LH}(\mathcal{E})^{\mathcal{I}^{\mathrm{op}}}$ which is essentially stable by subobjects. Let A be an object of $\mathcal{LH}(\mathcal{E})^{\mathcal{I}^{\mathrm{op}}}$. We know that there is a canonical epimorphism

$$\bigoplus_{i \in I} A(i)_i \longrightarrow A.$$

Since, for any $i \in \mathcal{I}$, there is an epimorphism

$$I(E(i)) \longrightarrow A(i)$$

with $E(i)$ in $\mathcal{E}$, we get an epimorphism

$$\bigoplus_{i \in I} I(E(i))_i \longrightarrow A.$$

By definition,

$$(\bigoplus_{i \in I} I(E(i))_i)(i') = \bigoplus_{i \in I} \bigoplus_{\alpha : i' \longrightarrow i} I(E(i)) \simeq I(\bigoplus_{i \in I} \bigoplus_{\alpha : i' \longrightarrow i} E(i))$$

Therefore,

$$\bigoplus_{i \in I} I(E(i))_i$$

is an object of $I(\mathcal{E}^{\mathcal{I}^{\mathrm{op}}})$. Applying Proposition 1.2.35, we get the conclusion. $\square$

1.4.3 Projective and inductive limits

Proposition 1.4.16. *Let $\mathcal{E}$ be a quasi-abelian category and let $\mathcal{I}$ be a small category. Assume $\mathcal{E}$ has exact products. Then,*

$$I(\varprojlim_{i\in\mathcal{I}} E(i)) \simeq \varprojlim_{i\in\mathcal{I}} I(E(i))$$

in $\mathcal{LH}(\mathcal{E})$. Moreover, a similar formula holds in $\mathcal{RH}(\mathcal{E})$ if and only if the functor

$$\varprojlim_{i\in\mathcal{I}} : \mathcal{E}^{\mathcal{I}} \to \mathcal{E}$$

is regular.

Proof. The first part follows directly from the fact that I has a left adjoint. Let us now consider the second part.
The condition is sufficient. Denote $\mathcal{J}$ the full subcategory of $\mathcal{E}^{\mathcal{I}}$ formed by the functors E for which the canonical morphism

$$I(\varprojlim_{i\in\mathcal{I}} E(i)) \to \varprojlim_{i\in\mathcal{I}} I(E(i))$$

is an isomorphism in $\mathcal{RH}(\mathcal{E})$. Thanks to Corollary 1.4.7

$$I : \mathcal{E} \to \mathcal{RH}(\mathcal{E})$$

preserves products. Hence, for any object F of $\mathcal{E}$ and any $i' \in \mathcal{I}$, we have

$$I(F^{i'}(i)) = I(F)^{i'}(i)$$

for any $i \in \mathcal{I}$ and it follows from Remark 1.4.13 that $F^{i'}$ is an object of $\mathcal{J}$. Since one checks also easily that a product of objects of $\mathcal{J}$ is in $\mathcal{J}$, it follows from Proposition 1.4.12 that any objet E of $\mathcal{E}^{\mathcal{I}}$ may be embedded in a strictly coexact sequence of the form

$$0 \to E \to J^0 \to J^1$$

where J^0 and J^1 are in $\mathcal{J}$. For such a sequence, the sequence

$$0 \to I \circ E \to I \circ J^0 \to I \circ J^1$$

is exact in $\mathcal{RH}(\mathcal{E})^{\mathcal{I}}$. Projective limit functors being strongly left exact, the sequence

$$0 \to \varprojlim_{i\in\mathcal{I}} I(E(i)) \to \varprojlim_{i\in\mathcal{I}} I(J^0(i)) \to \varprojlim_{i\in\mathcal{I}} I(J^1(i))$$

is exact in $\mathcal{RH}(\mathcal{E})$ and the sequence

$$0 \longrightarrow \varprojlim_{i\in\mathcal{I}} E(i) \longrightarrow \varprojlim_{i\in\mathcal{I}} J^0(i) \longrightarrow \varprojlim_{i\in\mathcal{I}} J^1(i)$$

is strictly exact in $\mathcal{E}$. Thanks to our assumption, the last morphism in this second sequence is strict. Hence the sequence

$$0 \longrightarrow I(\varprojlim_{i\in\mathcal{I}} E(i)) \longrightarrow I(\varprojlim_{i\in\mathcal{I}} J^0(i)) \longrightarrow I(\varprojlim_{i\in\mathcal{I}} J^1(i))$$

is exact in $\mathcal{RH}(\mathcal{E})$. The conclusion follows easily.
The condition is necessary. Let

$$f : E \longrightarrow F$$

be a strict morphism of $\mathcal{E}^{\mathcal{I}}$. Since the sequence

$$0 \longrightarrow \mathrm{Ker} f \longrightarrow E \longrightarrow F$$

is strictly coexact in $\mathcal{E}^{\mathcal{I}}$, the sequence

$$0 \longrightarrow I \circ \mathrm{Ker} f \longrightarrow I \circ E \longrightarrow I \circ F$$

is exact in $\mathcal{RH}(\mathcal{E})^{\mathcal{I}}$. Hence, the sequence

$$0 \longrightarrow \varprojlim_{i\in\mathcal{I}} I(\mathrm{Ker} f(i)) \longrightarrow \varprojlim_{i\in\mathcal{I}} I(E(i)) \longrightarrow \varprojlim_{i\in\mathcal{I}} I(F(i))$$

is exact in $\mathcal{RH}(\mathcal{E})$. Thanks to our assumption, it follows that the sequence

$$0 \longrightarrow I(\varprojlim_{i\in\mathcal{I}} \mathrm{Ker} f(i)) \longrightarrow I(\varprojlim_{i\in\mathcal{I}} E(i)) \longrightarrow I(\varprojlim_{i\in\mathcal{I}} F(i))$$

is also exact in $\mathcal{RH}(\mathcal{E})$. Hence, the sequence

$$0 \longrightarrow \varprojlim_{i\in\mathcal{I}} \mathrm{Ker} f(i) \longrightarrow \varprojlim_{i\in\mathcal{I}} E(i) \longrightarrow \varprojlim_{i\in\mathcal{I}} F(i)$$

is strictly coexact in $\mathcal{E}$ and the morphism

$$\varprojlim_{i\in\mathcal{I}} E(i) \longrightarrow \varprojlim_{i\in\mathcal{I}} F(i)$$

is strict. □

Proposition 1.4.17. *Let $\mathcal{E}$ be a cocomplete quasi-abelian category. Then filtering inductive limits are exact in $\mathcal{LH}(\mathcal{E})$ and commutes with*

$$I : \mathcal{E} \longrightarrow \mathcal{LH}(\mathcal{E})$$

if and only if filtering inductive limits are strongly exact in $\mathcal{E}$.

Proof. The condition is easily seen to be necessary, let us prove that it is also sufficient. A strongly exact functor being regular, the dual of the preceding proposition shows already that filtering inductive limits commute with I. To prove that filtering inductive limits are exact in $\mathcal{LH}(\mathcal{E})$, note that the functor

$$\varinjlim_{i\in\mathcal{I}} : \mathcal{E}^{\mathcal{I}} \longrightarrow \mathcal{E}$$

being strongly exact, it is strictly exact. Hence, by Proposition 1.2.33, it gives rise to an exact functor

$$L : \mathcal{LH}(\mathcal{E}^{\mathcal{I}}) \longrightarrow \mathcal{LH}(\mathcal{E})$$

and a canonical isomorphism

$$L \circ I_{\mathcal{E}^{\mathcal{I}}} \simeq I_{\mathcal{E}} \circ \varinjlim_{i\in\mathcal{I}} .$$

Composing L with the canonical equivalence

$$\mathcal{LH}(\mathcal{E})^{\mathcal{I}} \approx \mathcal{LH}(\mathcal{E}^{\mathcal{I}}) \tag{*}$$

we get an exact functor

$$L' : \mathcal{LH}(\mathcal{E})^{\mathcal{I}} \longrightarrow \mathcal{LH}(\mathcal{E}).$$

Let A be an object of $\mathcal{LH}(\mathcal{E})^{\mathcal{I}}$. Equivalence (*) shows that, in $\mathcal{LH}(\mathcal{E})^{\mathcal{I}}$, we have an exact sequence of the form

$$0 \longrightarrow I_{\mathcal{E}} \circ E_1 \xrightarrow{I(\delta)} I_{\mathcal{E}} \circ E_0 \longrightarrow A \longrightarrow 0$$

where

$$\delta : E_1 \longrightarrow E_0$$

is a monomorphism of $\mathcal{E}^{\mathcal{I}}$. It follows that the sequence

$$0 \longrightarrow L \circ I_{\mathcal{E}^{\mathcal{I}}}(E_1) \longrightarrow L \circ I_{\mathcal{E}^{\mathcal{I}}}(E_0) \longrightarrow L'(A) \longrightarrow 0$$

is exact in $\mathcal{LH}(\mathcal{E})$. Therefore, the sequence

$$0 \longrightarrow \varinjlim_{i\in\mathcal{I}} I_{\mathcal{E}}(E_1(i)) \longrightarrow \varinjlim_{i\in\mathcal{I}} I_{\mathcal{E}}(E_0(i)) \longrightarrow L'(A) \longrightarrow 0$$

is exact in $\mathcal{LH}(\mathcal{E})$. It follows that

$$L'(A) \simeq \varinjlim_{i\in\mathcal{I}} A(i)$$

and one checks easily that this isomorphism is both canonical and functorial. The conclusion follows directly. $\square$

1.5 Closed quasi-abelian categories

1.5.1 Closed structures, rings and modules

Recall (see e.g. [10]) that a closed additive category is an additive category $\mathcal{E}$ endowed with an internal tensor product

$$T : \mathcal{E} \times \mathcal{E} \to \mathcal{E},$$

a unit object

$$U \in \mathrm{Ob}(\mathcal{E}),$$

an internal homomorphism functor

$$H : \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{E}$$

and functorial isomorphisms

$$T(E, F) \simeq T(F, E)$$

$$T(U, E) \simeq E$$

$$\mathrm{Hom}_{\mathcal{E}}(T(E, F), G) \simeq \mathrm{Hom}_{\mathcal{E}}(E, H(F, G));$$

these data being subject to a few natural coherence axioms.

Let $\mathcal{E}$ be a closed additive category. By a ring in $\mathcal{E}$, we mean an unital monoid of $\mathcal{E}$. It corresponds to the data of an object R of $\mathcal{E}$, a multiplication morphism

$$m : T(R, R) \to R$$

and a unit morphism

$$u : U \to R.$$

These data being assumed to give rise to the usual commutative diagrams expressing that m is associative and that u is a unit for m.

Let R be a ring in $\mathcal{E}$. By an R-module, we mean an object E of $\mathcal{E}$ endowed with a (left) action of R. This action is a morphism

$$a : T(R, M) \to M$$

which gives rise to the usual commutative diagrams expressing its compatibility with the multiplication m and the unit u of R.

A morphism of an R-module E to an R-module F is defined as a morphism

$$f : E \to F$$

of $\mathcal{E}$ which is compatible with the actions of E and F. One checks easily that, with this definition of morphisms, R-modules form an additive category which we denote by $\mathcal{M}od(R)$.

We leave it to the reader to check the following result.

Proposition 1.5.1. *Let $\mathcal{E}$ be a closed additive category and let R be a ring of $\mathcal{E}$. Assume $\mathcal{E}$ is quasi-abelian (resp. abelian). Then, $\mathcal{M}od(R)$ is quasi-abelian (resp. abelian). Moreover, the forgetful functor*

$$\mathcal{M}od(R) \longrightarrow \mathcal{E}$$

preserves limits and colimits. In particular, a morphism of $\mathcal{M}od(R)$ is strict if and only if it is strict as a morphism of $\mathcal{E}$.

Proposition 1.5.2. *For any object E of $\mathcal{E}$, the multiplication m of R induces an action of R on*

$$T(R, E).$$

For any R-module F, we have

$$\operatorname{Hom}_{\mathcal{M}od(R)}(T(R, E), F) \simeq \operatorname{Hom}_{\mathcal{E}}(E, F).$$

In particular, $T(R, E)$ is projective in $\mathcal{M}od(R)$ if E is projective in $\mathcal{E}$.

Proof. The fact that $T(R, E)$ is an R-module is obvious. Let us prove the isomorphism. Define

$$\varphi : \operatorname{Hom}_{\mathcal{M}od(R)}(T(R, E), F) \longrightarrow \operatorname{Hom}_{\mathcal{E}}(E, F)$$

by setting

$$\varphi(h) = h \circ \alpha$$

where $\alpha : E \longrightarrow R \otimes E$ is the composition

$$E \simeq T(U, E) \xrightarrow{u \otimes \mathrm{id}_E} T(R, E)$$

and

$$\psi : \operatorname{Hom}_{\mathcal{E}}(E, F) \longrightarrow \operatorname{Hom}_{\mathcal{M}od(R)}(T(R, E), F)$$

by setting

$$\psi(h) = a_F \circ T(\mathrm{id}_R, h)$$

where a_F is the action of R on F. A simple computation shows that ψ is an inverse of φ and the conclusion follows. $\square$

1.5.2 Induced closed structure on $\mathcal{LH}(\mathcal{E})$

Proposition 1.5.3. *Let $\mathcal{E}$ be a closed quasi-abelian category with enough projective objects. Denote*

$$T : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$$

the internal tensor product, U the unit object and

$$H : \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{E}$$

the internal homomorphism functor. Assume that for any projective object P the functor $T(P, \cdot)$ is exact and that $T(P, P')$ is projective if P' is projective. Then,

(a) $H(P, \cdot)$ *is exact if P is projective,*

(b) $H(\cdot, I)$ *is exact if I is injective,*

(c) $H(P, I)$ *is injective if P is projective and I is injective.*

Moreover, T is explicitly left derivable, H is explicitly right derivable and we have the canonical functorial isomorphisms

(d) $LT(X, Y) \simeq LT(Y, X)$,

(e) $LT(X, U) \simeq X \simeq LT(U, X)$,

(f) $\mathrm{RHom}\,(LT(X, Y), Z) \simeq \mathrm{RHom}\,(X, RH(Y, Z))$,

(g) $RH(U, Z) \simeq Z$.

where X, $Y \in \mathcal{D}^-(\mathcal{E})$, $Z \in \mathcal{D}^+(\mathcal{E})$.

Proof. Thanks to our assumptions, (a), (b) and (c) follow directly from the adjunction formula

$$\mathrm{Hom}\,(T(X, Y), Z) = \mathrm{Hom}\,(X, H(Y, Z)).$$

Let $\mathcal{P}$ denote the full subcategory of $\mathcal{E}$ formed by projective objects. It follows from the hypothesis that $(\mathcal{P}, \mathcal{E})$ is T-projective and that $(\mathcal{P}^{\mathrm{op}}, \mathcal{E})$ is H-injective. Therefore, T is explicitly left derivable and H is explicitly right derivable. To prove (d), (e) and (f), we may reduce to the case where X, Y are objects of $\mathcal{K}^-(\mathcal{P})$. In this case, $LT(X, Y) \simeq T(X, Y)$ and $RH(Y, Z) = H(Y, Z)$. Since $T(X, Y)$ is an object of $\mathcal{K}^-(\mathcal{P})$, everything follows from the fact that $\mathcal{E}$ is a closed quasi-abelian category. To prove (g), we use (f) with $Y = U$. This gives us the isomorphism

$$\mathrm{RHom}\,(X, Z) \simeq \mathrm{RHom}\,(X, RH(U, Z))$$

where $X \in \mathcal{D}^-(\mathcal{E})$, $Z \in \mathcal{D}^+(\mathcal{E})$. Fix $Z \in \mathcal{D}^+(\mathcal{E})$ and denote C the cone of the canonical morphism

$$Z \longrightarrow H(U, Z) \longrightarrow RH(U, Z).$$

It follows from what precedes that

$$\mathrm{RHom}\,(X, C) \simeq 0$$

for any $X \in \mathcal{D}^-(\mathcal{E})$. Hence, the complex

$$\mathrm{Hom}\,(X, C)$$

is exact for any $X \in \mathcal{P}$ and C itself is strictly exact (see Proposition 1.3.23). Therefore, $C \simeq 0$ and

$$Z \simeq RH(U, Z).$$

□

Corollary 1.5.4. *In the situation of the preceding proposition, $\mathcal{LH}(\mathcal{E})$ is canonically a closed abelian category. Its internal tensor product is given by*

$$\tilde{T} = LH^0 \circ LT : \mathcal{LH}(\mathcal{E}) \times \mathcal{LH}(\mathcal{E}) \longrightarrow \mathcal{LH}(\mathcal{E}),$$

its unit object by $\tilde{U} = I(U)$ and its internal homomorphism functor by

$$\tilde{H} = LH^0 \circ RH : \mathcal{LH}(\mathcal{E})^{\mathrm{op}} \times \mathcal{LH}(\mathcal{E}) \longrightarrow \mathcal{LH}(\mathcal{E}).$$

The functor $\tilde{T}$ (resp. $\tilde{H}$) is explicitly left (resp. right) derivable and we have the canonical isomorphisms

$$L\tilde{T}(I(X), I(Y)) \simeq LT(X, Y)$$

and

$$R\tilde{H}(I(Y), I(Z)) \simeq RH(Y, Z)$$

for any X, $Y \in \mathcal{D}^-(\mathcal{E})$ and any $Z \in \mathcal{D}^+(\mathcal{E})$.

Assume moreover that the functor

$$T(P, \cdot) : \mathcal{E} \longrightarrow \mathcal{E}$$

is strongly exact for any projective object P of $\mathcal{E}$. Then, for any projective object Q of $\mathcal{LH}(\mathcal{E})$ the functor

$$\tilde{T}(Q, \cdot) : \mathcal{LH}(\mathcal{E}) \longrightarrow \mathcal{LH}(\mathcal{E})$$

is exact and $\tilde{T}(Q, Q')$ is projective if Q' is projective in $\mathcal{LH}(\mathcal{E})$.

Proof. It follows from parts (d) and (e) of the preceding proposition that $\tilde{T}$ is symmetric and that $I(U)$ is a unit. The coherence axioms are also easily checked.

Let Y, $Z \in \mathcal{LH}(\mathcal{E})$. We know that Y is isomorphic to a complex P of $\mathcal{K}^-(\mathcal{P})$. Therefore,

$$RH(Y, Z) \simeq H(P, Z).$$

The first non-zero component of $H(P, Z)$ is of degree -1 and is isomorphic to

$$H(P^0, Z^{-1}),$$

the second non-zero component is of degree 0 and isomorphic to

$$\mathrm{Hom}\,(P^0, Z^0) \oplus \mathrm{Hom}\,(P^{-1}, Z^{-1})$$

the differential being

$$\begin{pmatrix} \mathrm{Hom}\,(P^0, d_Z^{-1}) \\ \mathrm{Hom}\,(d_P^{-1}, Z^{-1}) \end{pmatrix}.$$

Since $Z \in \mathcal{LH}(\mathcal{E})$, d_Z^{-1} is a monomorphism. So, the differential of degree -1 of $H(P, Z)$ is also a monomorphism and $RH(Y, Z) \in \mathcal{D}^{\geq 0}(\mathcal{E})$ for the left t-structure. Moreover, we have also $LT(X, Y) \in \mathcal{D}^{\leq 0}(\mathcal{E})$ for any X, $Y \in \mathcal{LH}(\mathcal{E})$. Therefore, using part (e) of the preceding proposition, we get successively

$$\begin{aligned}
\mathrm{Hom}\,_{\mathcal{LH}(\mathcal{E})}(\tilde{T}(X, Y), Z) &\simeq \mathrm{Hom}\,_{\mathcal{D}^{\geq 0}(\mathcal{E})}(\tau^{\geq 0} \circ LT(X, Y), Z) \\
&\simeq \mathrm{Hom}\,_{\mathcal{D}(\mathcal{E})}(LT(X, Y), Z) \\
&\simeq \mathrm{Hom}\,_{\mathcal{D}(\mathcal{E})}(X, RH(Y, Z)) \\
&\simeq \mathrm{Hom}\,_{\mathcal{D}^{\leq 0}(\mathcal{E})}(X, \tau^{\leq 0} \circ RH(Y, Z)) \\
&\simeq \mathrm{Hom}\,_{\mathcal{LH}(\mathcal{E})}(X, \tilde{H}(Y, Z))
\end{aligned}$$

for X, Y, $Z \in \mathcal{LH}(\mathcal{E})$.

Since $\mathcal{LH}(\mathcal{E})$ has enough projective objects, $\tilde{T}$ is clearly left derivable.

Consider $P \in \mathcal{P}$ and an exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

of $\mathcal{LH}(\mathcal{E})$. This sequence corresponds to a distinguished triangle

$$A' \longrightarrow A \longrightarrow A'' \xrightarrow{+1}$$

of $\mathcal{D}^+(\mathcal{E})$. Hence,

$$RH(P, A') \longrightarrow RH(P, A) \longrightarrow RH(P, A'') \xrightarrow{+1}$$

is a distinguished triangle of $\mathcal{D}^+(\mathcal{E})$. Since $H(P,\cdot)$ is strongly left exact,

$$RH(P,Y) \simeq H(P,Y)$$

is in $\mathcal{LH}(\mathcal{E})$ when Y is replaced by A', A or A''. Therefore, the sequence

$$0 \to \tilde{H}(I(P),A') \to \tilde{H}(I(P),A) \to \tilde{H}(I(P),A'') \to 0$$

is exact in $\mathcal{LH}(\mathcal{E})$. It follows that $(I(\mathcal{P}),\mathcal{LH}(\mathcal{E}))$ is $\tilde{H}$-injective and that $\tilde{H}$ is explicitly right derivable.

To conclude, it is sufficient to recall that any object X of $\mathcal{D}^-(\mathcal{E})$ is isomorphic to the image of an object of $\mathcal{K}^-(\mathcal{P})$ and to note that if X, $Y \in \mathcal{P}$ and $Z \in \mathcal{E}$, we have

$$\begin{aligned} L\tilde{T}(I(X),I(Y)) &\simeq \tilde{T}(I(X),I(Y)) \\ &\simeq LH^0 \circ LT(X,Y) \\ &\simeq T(X,Y) \end{aligned}$$

and

$$\begin{aligned} R\tilde{H}(I(Y),I(Z)) &\simeq \tilde{H}(I(Y),I(Z)) \\ &\simeq LH^0 \circ RH(Y,Z) \\ &\simeq H(Y,Z). \end{aligned}$$

Let us now treat the last part of the statement. Thanks to Proposition 1.3.24, we may assume that $Q = I(P)$ and that $Q' = I(P')$ where P and P' are projective objects of $\mathcal{E}$. Let

$$0 \to A' \to A \to A'' \to 0$$

be an exact sequence of $\mathcal{LH}(\mathcal{E})$. This corresponds to a distinguished triangle

$$A' \to A \to A'' \xrightarrow{+1}$$

of $\mathcal{D}^-(\mathcal{E})$. Applying $LT(I(P),\cdot)$, we get the distinguished triangle

$$LT(I(P),A') \to LT(I(P),A) \to LT(I(P),A'') \xrightarrow{+1}$$

We know that

$$LT(I(P),X) \simeq T(P,X)$$

for any object X of $\mathcal{D}^-(\mathcal{E})$. Therefore, the long exact sequence of cohomology shows that $\tilde{T}(I(P),.)$ is exact on $\mathcal{LH}(\mathcal{E})$ if

$$LH^{-1}(T(P,A'')) \simeq 0$$

for any object A'' of $\mathcal{LH}(\mathcal{E})$. This will clearly be the case if

$$T(P, \cdot) : \mathcal{E} \longrightarrow \mathcal{E}$$

preserves monomorphisms. Keeping in mind the fact that $T(P, \cdot)$ is exact and strongly right exact, this last condition is equivalent to the one in our statement. To conclude we only have to note that

$$\tilde{T}(I(P), I(P')) \simeq I(T(P, P'))$$

and use Proposition 1.3.24. $\square$

Chapter 2

Sheaves with Values in Quasi-Abelian Categories

2.1 Elementary quasi-abelian categories

2.1.1 Small and tiny objects

In this subsection, we study various notions of smallness for an object of a quasi-abelian category. We leave it to the reader to state the corresponding results for the dual notions.

Definition 2.1.1. An object E of a cocomplete additive category $\mathcal{E}$ is

(a) *small* if

$$\operatorname{Hom}(E, \bigoplus_{i \in I} F_i) \simeq \bigoplus_{i \in I} \operatorname{Hom}(E, F_i),$$

for any small family $(F_i)_{i \in I}$ of $\mathcal{E}$.

(b) *tiny* if

$$\varinjlim_{i \in \mathcal{I}} \operatorname{Hom}(E, F(i)) \simeq \operatorname{Hom}(E, \varinjlim_{i \in \mathcal{I}} F(i))$$

for any filtering inductive system

$$E : \mathcal{I} \to \mathcal{E}.$$

Remark 2.1.2. Here, we have followed Grothendieck's definition of a small object. We don't know if the notion of a tiny object was already defined before. Of course, every tiny object is small. It is also easy to see that in a abelian category, every small projective object is tiny but this is not necessarily the case in a quasi-abelian one. There is also a possible stronger condition of smallness. This condition is clarified in the following proposition.

Proposition 2.1.3. *Let $\mathcal{E}$ be a cocomplete quasi-abelian category. An object E of $\mathcal{E}$ is such that*

$$\mathrm{Hom}\,(E, \varinjlim_{j\in\mathcal{J}} F(j)) \simeq \varinjlim_{j\in\mathcal{J}} \mathrm{Hom}\,(E, F(j))$$

for any inductive system

$$F : \mathcal{J} \to \mathcal{E}$$

if and only if E is a small strongly projective object of $\mathcal{E}$.

Proof. Taking for $\mathcal{I}$ a discrete category or a category of the form

$$\begin{tikzcd} \cdot \arrow[loop left] \arrow[r, bend left] \arrow[r, bend right] & \cdot \arrow[loop right] \end{tikzcd}$$

one sees that

$$\mathrm{Hom}\,(E, .)$$

preserves direct sums and cokernels. Hence, E is a small strongly projective object of $\mathcal{E}$.

Conversely, if E is a small strongly projective object, $\mathrm{Hom}\,(E, .)$ is both cokernel and direct sum preserving. Viewing

$$\varinjlim_{j\in\mathcal{J}} F(j)$$

as the cokernel of the canonical morphism

$$\bigoplus_{\alpha:j\longrightarrow j'} F(j) \to \bigoplus_{j\in\mathcal{J}} F(j)$$

allows us to conclude. □

Proposition 2.1.4. *Let $\mathcal{E}$ be a cocomplete quasi-abelian category. Assume filtering inductive limits are exact in $\mathcal{E}$. Then, a small projective object of $\mathcal{E}$ is tiny.*

Proof. Let $\mathcal{I}$ be a small filtering category and let P be a small projective object of $\mathcal{E}$. Denote $\mathcal{Q}$ the full subcategory of $\mathcal{E}^{\mathcal{I}}$ formed by the functors E for which the canonical morphism

$$\varinjlim_{i\in\mathcal{I}} \mathrm{Hom}\,(P, E(i)) \longrightarrow \mathrm{Hom}\,(P, \varinjlim_{i\in\mathcal{I}} E(i))$$

is an isomorphism. Since P is small, it is clear that $\mathcal{Q}$ is stable by direct sums. Moreover, for any E in $\mathcal{E}$ and any $i \in \mathcal{I}$, we have

$$\mathrm{Hom}\,(P, E_i(i')) \simeq \mathrm{Hom}\,(P, \bigoplus_{\alpha:i\longrightarrow i'} E) \simeq \bigoplus_{\alpha:i\longrightarrow i'} \mathrm{Hom}\,(P, E) \simeq \mathrm{Hom}\,(P, E)_i(i')$$

and it follows from the dual of Remark 1.4.13 that the functor E_i belongs to $\mathcal{Q}$. Therefore, using Proposition 1.4.12, we see that any object E of $\mathcal{E}^{\mathcal{I}}$ may be embedded in a strictly exact sequence of the form

$$Q_1 \longrightarrow Q_0 \longrightarrow E \longrightarrow 0$$

where Q_1 and Q_0 belong to $\mathcal{Q}$. Since filtering inductive limits are exact in $\mathcal{E}$, the sequence

$$\varinjlim_{i \in \mathcal{I}} Q_1(i) \longrightarrow \varinjlim_{i \in \mathcal{I}} Q_0(i) \longrightarrow \varinjlim_{i \in \mathcal{I}} E(i) \longrightarrow 0$$

is strictly exact in $\mathcal{E}$. Since P is projective in $\mathcal{E}$, we see that the sequences

$$\mathrm{Hom}\,(P, Q_1(i)) \longrightarrow \mathrm{Hom}\,(P, Q_0(i)) \longrightarrow \mathrm{Hom}\,(P, E(i)) \longrightarrow 0 \qquad (i \in \mathcal{I})$$

$$\mathrm{Hom}\,(P, \varinjlim_{i \in \mathcal{I}} Q_1(i)) \longrightarrow \mathrm{Hom}\,(P, \varinjlim_{i \in \mathcal{I}} Q_0(i)) \longrightarrow \mathrm{Hom}\,(P, \varinjlim_{i \in \mathcal{I}} E(i)) \longrightarrow 0$$

are exact. Inductive limits being exact in the category of abelian groups, it follows that

$$\varinjlim_{i \in \mathcal{I}} \mathrm{Hom}\,(P, E(i)) \simeq \mathrm{Hom}\,(P, \varinjlim_{i \in \mathcal{I}} E(i)).$$

□

2.1.2 Generating and strictly generating sets

Although, in this subsection, we consider only generating and strictly generating sets, the reader will easily obtain by duality similar considerations for cogenerating and strictly cogenerating sets.

Let us recall (see e.g. [10]) that a subset $\mathcal{G}$ of $\mathrm{Ob}(\mathcal{E})$ is a generating set of $\mathcal{E}$ if for any pair

$$E \underset{f'}{\overset{f}{\rightrightarrows}} F$$

of distinct parallel morphisms of $\mathcal{E}$, there is a morphism

$$G \xrightarrow{e} E$$

with $G \in \mathcal{G}$, such that

$$f \circ e \neq f' \circ e.$$

It is clearly equivalent to ask that for any strict monomorphism $s : S \longrightarrow E$ which is not an isomorphism, there is a morphism

$$G \longrightarrow E$$

with $G \in \mathcal{G}$ which does not factor through s. Moreover, if $\mathcal{E}$ is cocomplete and $\mathcal{G}$ is small, it is also equivalent to ask that for any object E of $\mathcal{E}$ there is an epimorphism of the form

$$\bigoplus_{j \in J} G_j \longrightarrow E$$

where $(G_j)_{j \in J}$ is a small family of elements of $\mathcal{G}$

The preceding notion is suitable for the study of abelian categories where any monomorphism is strict. For quasi-abelian ones, the following definition is more useful.

Definition 2.1.5. A *strictly generating set* of $\mathcal{E}$ is a subset $\mathcal{G}$ of $\mathrm{Ob}(\mathcal{E})$ such that for any monomorphism

$$m : S \longrightarrow E$$

of $\mathcal{E}$ which is not an isomorphism, there is a morphism

$$G \longrightarrow E$$

with $G \in \mathcal{G}$ which does not factor through m.

Lemma 2.1.6. *Let $u : E \longrightarrow F$ be a morphism of $\mathcal{E}$. Then, the following conditions are equivalent:*

(a) u is a strict epimorphism,

(b) u does not factor through a monomorphism $s : S \longrightarrow E$ which is not an isomorphism.

Proof.
(a) $\Rightarrow$ (b). Let $s : S \longrightarrow E$ be a monomorphism. Assume $u' : S \longrightarrow E$ is such that

$$u = s \circ u'.$$

Since u is a strict epimorphism, so is s. Hence, s is an isomorphism and the conclusion follows.
(b) $\Rightarrow$ (a). Since u factors through the monomorphism $\mathrm{Coim} u \longrightarrow E$,

$$\mathrm{Coim} u \simeq E$$

and u is a strict epimorphism.

□

Proposition 2.1.7. *Let $\mathcal{E}$ be a cocomplete quasi-abelian category. A small subset $\mathcal{G}$ of $\mathrm{Ob}(\mathcal{E})$ is a strictly generating set of $\mathcal{E}$ if and only if for any object E of $\mathcal{E}$, there is a strict epimorphism of the form*

$$\bigoplus_{j\in J} G_j \to E$$

where $(G_j)_{j\in J}$ is a small family of elements of $\mathcal{G}$.

Proof. Assume $\mathcal{G}$ is a strictly generating set of $\mathcal{E}$. Consider the canonical morphism

$$\bigoplus_{G\in\mathcal{G},h\in\mathrm{Hom}\,(G,E)} G \xrightarrow{e} E.$$

Using the preceding lemma, let us prove by contradiction that e is a strict epimorphism. Let

$$m : S \to E$$

be a monomorphism which is not an isomorphism and assume $e = m \circ f$ for some $f : X \to S$. For any $G \in \mathcal{G}$ and any $h \in \mathrm{Hom}\,(G, E)$, we get

$$h = m \circ f \circ i_{(G,h)}.$$

in contradiction with Definition 2.1.5.

Conversely, assume we have a strict epimorphism

$$\bigoplus_{j\in J} G_j \xrightarrow{h} E$$

and let $m : S \to E$ be a monomorphism which is not an isomorphism. Assume that for any $j \in J$

$$h \circ s_j : G_j \to E$$

factors through m. This gives us a family of morphisms

$$h'_j : G_j \to S \qquad (j \in J)$$

such that

$$h \circ s_j = m \circ h'_j.$$

Consider the morphism

$$h' : \bigoplus_{j\in J} G_j \to S$$

associated to the family $(h'_j)_{j\in J}$. Clearly,

$$m \circ h' = h.$$

This contradicts the fact that h is a strict epimorphism. Therefore, one of the

$$h \circ s_j$$

does not factor through m and $\mathcal{G}$ is a strict generating set of $\mathcal{E}$. $\square$

Proposition 2.1.8. *Let $\mathcal{G}$ be a small strictly generating set of the cocomplete quasi-abelian category $\mathcal{E}$. Then, a sequence*

$$0 \longrightarrow E' \xrightarrow{e'} E \xrightarrow{e''} E''$$

is strictly exact in $\mathcal{E}$ if and only if the sequence of abelian groups

$$0 \longrightarrow \operatorname{Hom}(G, E') \longrightarrow \operatorname{Hom}(G, E) \longrightarrow \operatorname{Hom}(G, E'')$$

is exact for any $G \in \mathcal{G}$.

Assume moreover that the elements of $\mathcal{G}$ are projective objects of $\mathcal{E}$. Then, a sequence

$$E' \xrightarrow{e'} E \xrightarrow{e''} E''$$

is strictly exact in $\mathcal{E}$ if and only if the sequence of abelian groups

$$\operatorname{Hom}(G, E') \longrightarrow \operatorname{Hom}(G, E) \longrightarrow \operatorname{Hom}(G, E'')$$

is exact for any $G \in \mathcal{G}$.

Proof. Proceed as in the proof of Proposition 1.3.23. $\square$

Proposition 2.1.9. *Let $\mathcal{E}$ be a cocomplete quasi-abelian category. Assume $\mathcal{E}$ has a small strictly generating set $\mathcal{G}$ of small (resp. tiny) objects. Then, direct sums (resp. filtering inductive limits) are strongly exact in $\mathcal{E}$.*

Proof. Let

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E''$$

be a strict exact sequence of $\mathcal{E}^{\mathcal{I}}$, where $\mathcal{I}$ is a small discrete (resp. filtering) category. For any $i \in \mathcal{I}$, the sequence

$$0 \longrightarrow E'(i) \longrightarrow E(i) \longrightarrow E''(i)$$

is strictly exact in $\mathcal{E}$. Hence, for any $G \in \mathcal{G}$, we get the exact sequence

$$0 \longrightarrow \operatorname{Hom}(G, E'(i)) \longrightarrow \operatorname{Hom}(G, E(i)) \longrightarrow \operatorname{Hom}(G, E''(i))$$

By taking the inductive limit and using the fact that G is small (resp. tiny), we get the exact sequence

$$0 \longrightarrow \operatorname{Hom}(G, \varinjlim_{i \in \mathcal{I}} E'(i)) \longrightarrow \operatorname{Hom}(G, \varinjlim_{i \in \mathcal{I}} E(i)) \longrightarrow \operatorname{Hom}(G, \varinjlim_{i \in \mathcal{I}} E''(i))$$

The conclusion follows easily from Proposition 2.1.8. $\square$

2.1.3 Quasi-elementary and elementary categories

Definition 2.1.10. A quasi-abelian category is *quasi-elementary* (resp. *elementary*) if it is cocomplete and has a small strictly generating set of small (resp. tiny) projective objects.

Remark 2.1.11. One checks easily that an abelian category is elementary if and only if it is quasi-elementary. So, the preceding definition is compatible with the definition of elementary abelian categories in [14].

Proposition 2.1.12. *A quasi-abelian category $\mathcal{E}$ is quasi-elementary if and only if $\mathcal{LH}(\mathcal{E})$ is elementary.*

Proof. Let us prove that the condition is necessary. Thanks to the dual of Proposition 2.1.9, direct sums are strongly exact in $\mathcal{E}$. Therefore, the dual of Proposition 1.4.7 shows that $\mathcal{LH}(\mathcal{E})$ is cocomplete and that

$$I : \mathcal{E} \to \mathcal{LH}(\mathcal{E})$$

preserves direct sums. Let $\mathcal{P}$ be a strictly generating small set of small projective objects of $\mathcal{E}$. Clearly, $I(\mathcal{P})$ is a generating small set of $\mathcal{LH}(\mathcal{E})$ and Proposition 1.3.24 shows that the objects of $I(\mathcal{P})$ are projective in $\mathcal{LH}(\mathcal{E})$. To check that $I(P)$ is small for any object P of $\mathcal{P}$, we may proceed as follows. Let $(A_i)_{i\in I}$ be a family of $\mathcal{LH}(\mathcal{E})$. We get a family of short exact sequences of $\mathcal{LH}(\mathcal{E})$

$$0 \to I(F_i) \to I(E_i) \to A_i \to 0.$$

Therefore, the sequence

$$\bigoplus_{i\in I} I(F_i) \to \bigoplus_{i\in I} I(E_i) \to \bigoplus_{i\in I} A_i \to 0$$

is exact in $\mathcal{LH}(\mathcal{E})$ and the sequence

$$\mathrm{Hom}\,(I(P), \bigoplus_{i\in I} I(F_i)) \to \mathrm{Hom}\,(I(P), \bigoplus_{i\in I} I(E_i)) \to \mathrm{Hom}\,(I(P), \bigoplus_{i\in I} A_i) \to 0$$

is exact in $\mathcal{A}b$. Since

$$\bigoplus_{i\in I} I(E_i) = I(\bigoplus_{i\in I} E_i)$$

and P is small, we get the exact sequence

$$\bigoplus_{i\in I} \mathrm{Hom}\,(P, F_i) \to \bigoplus_{i\in I} \mathrm{Hom}\,(P, E_i) \to \mathrm{Hom}\,(I(P), \bigoplus_{i\in I} A_i) \to 0.$$

It follows that

$$\bigoplus_{i\in I} \operatorname{Hom}(I(P), A_i) \simeq \operatorname{Hom}(I(P), \bigoplus_{i\in I} A_i)$$

and we see that $I(P)$ is small in $\mathcal{LH}(\mathcal{E})$.

The condition is also sufficient. Since $\mathcal{LH}(\mathcal{E})$ is cocomplete, it follows from the dual of Proposition 1.4.8 that $\mathcal{E}$ is cocomplete. Let $\mathcal{P}$ be a generating small set of small projective objects of $\mathcal{LH}(\mathcal{E})$. It follows that $C(\mathcal{P})$ is a generating small set of projective objects of $\mathcal{E}$. To check that $C(\mathcal{P})$ is small for any object P of $\mathcal{P}$, we can proceed as follows. Since the canonical morphism

$$\bigoplus_{i\in I} I(E_i) \to I(\bigoplus_{i\in I} E_i)$$

is epimorphic, we get an epimorphism

$$\operatorname{Hom}(P, \bigoplus_{i\in I} I(E_i)) \to \operatorname{Hom}(P, I(\bigoplus_{i\in I} E_i)).$$

Since P is small, we see that the canonical morphism

$$\bigoplus_{i\in I} \operatorname{Hom}(P, I(E_i)) \to \operatorname{Hom}(P, I(\bigoplus_{i\in I} E_i))$$

is surjective. By adjunction, it follows that the canonical morphism

$$\bigoplus_{i\in I} \operatorname{Hom}(C(P), E_i) \to \operatorname{Hom}(C(P), \bigoplus_{i\in I} E_i)$$

is surjective. Since it is also clearly injective, $C(P)$ is small. $\square$

Remark 2.1.13. Thanks to the preceding proposition, we may reduce the following Proposition to Freyd's characterization of functor categories. However, for the reader's convenience, we give a direct proof.

Proposition 2.1.14. *Let $\mathcal{E}$ be a cocomplete quasi-abelian category and let $\mathcal{P}$ be a full additive subcategory of $\mathcal{E}$. Assume that the objects of $\mathcal{P}$ form a strictly generating small set of small projective objects of $\mathcal{E}$. Then, the functor*

$$\begin{aligned} h : \mathcal{E} &\to \mathcal{A}dd(\mathcal{P}^{\mathrm{op}}, \mathcal{A}b) \\ E &\mapsto \operatorname{Hom}_{\mathcal{E}}(., E) \end{aligned}$$

is strictly exact and induces an equivalence of categories

$$\mathcal{LH}(\mathcal{E}) \approx \mathcal{A}dd(\mathcal{P}^{\mathrm{op}}, \mathcal{A}b).$$

Proof. For any strictly exact sequence

$$E' \xrightarrow{f} E \xrightarrow{g} E''$$

of $\mathcal{E}$ and any object P of $\mathcal{P}$, the sequence

$$\operatorname{Hom}_{\mathcal{E}}(P, E') \longrightarrow \operatorname{Hom}_{\mathcal{E}}(P, E) \longrightarrow \operatorname{Hom}_{\mathcal{E}}(P, E'')$$

is exact since P is projective. Hence, the functor h is strictly exact and induces a functor

$$h : \mathcal{D}^{-}(\mathcal{E}) \longrightarrow \mathcal{D}^{-}(\mathcal{A}dd(\mathcal{P}^{\mathrm{op}}, \mathcal{A}b)).$$

Consider the category $\mathcal{L}$ whose objects are defined by

$$\mathrm{Ob}(\mathcal{L}) = \{(P_i)_{i\in I} : I \text{ small set},\ P_i \in \mathrm{Ob}(\mathcal{P})\}$$

and whose morphisms are defined by

$$\operatorname{Hom}_{\mathcal{L}}((P_i)_{i\in I}, (P'_j)_{j\in J}) = \prod_{i\in I} \bigoplus_{j\in J} \operatorname{Hom}_{\mathcal{E}}(P_i, P'_j).$$

So, a morphism f of

$$\operatorname{Hom}_{\mathcal{L}}((P_i)_{i\in I}, (P'_j)_{j\in J})$$

may be considered as an infinite matrix (f_{ji}) with

$$f_{ji} : P_i \longrightarrow P'_j \qquad \forall i \in I, \quad \forall j \in J.$$

the set

$$\{j : f_{ji} \neq 0\}$$

being finite for any $i \in I$.

Now, let us define the functor

$$S : \mathcal{L} \longrightarrow \mathcal{E}$$

by setting

$$S((P_i)_{i\in I}) = \bigoplus_{i\in I} P_i$$

for any object $(P_i)_{i\in I}$ of $\mathcal{L}$. For any morphism

$$f : (P_i)_{i\in I} \longrightarrow (P'_j)_{j\in J}$$

we define

$$S(f) : \bigoplus_{i\in I} P_i \longrightarrow \bigoplus_{j\in J} P'_j.$$

by setting

$$S(f) \circ s_i = \sum_{j \in J} s'_j \circ f_{ji}$$

where

$$s_i : P_i \longrightarrow \bigoplus_{i \in I} P_i, \quad s'_j : P'_j \longrightarrow \bigoplus_{j \in J} P'_j$$

are the canonical morphisms.

Since P_i is small for any $i \in I$, the functor S is fully faithful. As a matter of fact, for any objects $(P_i)_{i \in I}$ and $(P'_j)_{j \in J}$ of $\mathcal{L}$, we have successively

$$\begin{aligned}\operatorname{Hom}_{\mathcal{E}}(S((P_i)_{i\in I}), S((P'_j)_{j\in J})) &\simeq \operatorname{Hom}_{\mathcal{E}}(\bigoplus_{i\in I} P_i, \bigoplus_{j\in J} P'_j) \\ &\simeq \prod_{i\in I} \bigoplus_{j\in J} \operatorname{Hom}_{\mathcal{E}}(P_i, P'_j) \\ &\simeq \operatorname{Hom}_{\mathcal{L}}((P_i)_{i\in I}, (P'_j)_{j\in J}).\end{aligned}$$

Hence, $\mathcal{L}$ is equivalent to a full subcategory $S(\mathcal{L})$ of $\mathcal{E}$. Since the direct sum of a family of projective objects is a projective object, the objects of $S(\mathcal{L})$ are projective. Moreover, by hypothesis, for any object E of $\mathcal{E}$, there is a strict epimorphism

$$S((P_i)_{i\in I}) \longrightarrow E$$

where $(P_i)_{i\in I}$ is an object of $\mathcal{L}$. Therefore, by Proposition 1.3.22 we have an equivalence of categories

$$\mathcal{K}^-(S(\mathcal{L})) \approx \mathcal{D}^-(\mathcal{E}).$$

and the functor S induces an equivalence of categories

$$\mathcal{K}^-(\mathcal{L}) \approx \mathcal{D}^-(\mathcal{E}).$$

Let P be an object of $\mathcal{P}$. Recall that the functor

$$h^P : \mathcal{P}^{\mathrm{op}} \longrightarrow \mathcal{A}b$$

is a small projective object of $\mathcal{A}dd(\mathcal{P}^{\mathrm{op}}, \mathcal{A}b)$. As a matter of fact, for any object F of $\mathcal{A}dd(\mathcal{P}^{\mathrm{op}}, \mathcal{A}b)$, we have

$$\operatorname{Hom}(h^P, F) \simeq F(P).$$

Hence, for any family $(F)_{i\in I}$ of $\mathcal{A}dd(\mathcal{P}^{\mathrm{op}}, \mathcal{A}b)$ we get

$$\begin{aligned}\operatorname{Hom}(h^P, \bigoplus_{i\in I} F_i) &\simeq (\bigoplus_{i\in I} F_i)(P) \\ &\simeq \bigoplus_{i\in I} F_i(P) \\ &\simeq \bigoplus_{i\in I} \operatorname{Hom}(h^P, F_i),\end{aligned}$$

and h^P is small. Moreover, if

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

is an exact sequence of $\mathcal{A}dd(\mathcal{P}^{\mathrm{op}}, \mathcal{A}b)$ then the sequence

$$0 \longrightarrow F'(P) \longrightarrow F(P) \longrightarrow F''(P) \longrightarrow 0$$

is exact. Therefore, the sequence

$$0 \longrightarrow \operatorname{Hom}(h^P, F') \longrightarrow \operatorname{Hom}(h^P, F) \longrightarrow \operatorname{Hom}(h^P, F'') \longrightarrow 0$$

is also exact and h^P is projective.

For any object F of $\mathcal{A}dd(\mathcal{P}^{\mathrm{op}}, \mathcal{A}b)$, we define the morphism

$$v : \bigoplus_{\{(P,f):P\in\mathcal{P},\, f\in F(P)\}} h^P \longrightarrow F$$

by setting

$$v \circ s_{(P,f)} = \psi_P(f)$$

where

$$\psi_P(f) : h^P \longrightarrow F$$

is defined by

$$\psi_P(f)(P')(g) = F(g)(f)$$

for any object P' of $\mathcal{P}$ and any morphism g of $\operatorname{Hom}_{\mathcal{E}}(P', P)$. Let us show that v is an epimorphism. It is sufficient to show that for any object P' of $\mathcal{P}$ the morphism

$$v(P') : \bigoplus_{\{(P,f):P\in\mathcal{P},\, f\in F(P)\}} h^P(P') \longrightarrow F(P')$$

is surjective. Consider $f' \in F(P')$. Since $\mathrm{id}_{P'} \in h^{P'}(P') = \operatorname{Hom}_{\mathcal{E}}(P', P')$, we have

$$\begin{aligned} v(s_{(P',f')}(P')(\mathrm{id}_{P'})) &= \psi_{P'}(f')(P')(\mathrm{id}_{P'}) \\ &= F(\mathrm{id}_{P'})(f') \\ &= f' \end{aligned}$$

and the conclusion follows.

Let

$$S' : \mathcal{L} \longrightarrow \mathcal{A}dd(\mathcal{P}^{\mathrm{op}}, \mathcal{A}b)$$

be the functor defined by setting

$$S'((P_i)_{i\in I}) = \bigoplus_{i\in I} h^{P_i}.$$

Thanks to the preceding discussion, we may apply to S' the same kind of arguments we applied to S and conclude that S' induces an equivalence of categories

$$S' : \mathcal{K}^-(\mathcal{L}) \xrightarrow{\sim} \mathcal{D}^-(\mathcal{A}dd(\mathcal{P}^{\mathrm{op}}, \mathcal{A}b)).$$

Moreover, since in the diagram

$$\begin{tikzcd} \mathcal{D}^-(\mathcal{E}) \arrow[rr, "h"] & & \mathcal{D}^-(\mathcal{A}dd(\mathcal{P}^{\mathrm{op}}, \mathcal{A}b)) \\ & \mathcal{K}^-(\mathcal{L}) \arrow[ul, "S"] \arrow[ur, "S'"'] & \end{tikzcd}$$

we have clearly $h \circ S = S'$, it follows that

$$h : \mathcal{D}^-(\mathcal{E}) \longrightarrow \mathcal{D}^-(\mathcal{A}dd(\mathcal{P}^{\mathrm{op}}, \mathcal{A}b))$$

is also an equivalence of categories.

Using the fact that h is strictly exact, we see that it exchanges the hearts of $\mathcal{D}^-(\mathcal{E})$ and $\mathcal{D}^-(\mathcal{A}dd(\mathcal{P}^{\mathrm{op}}, \mathcal{A}b))$ corresponding left t-structures. So, h induces an equivalence of categories

$$h : \mathcal{LH}(\mathcal{E}) \longrightarrow \mathcal{LH}(\mathcal{A}dd(\mathcal{P}^{\mathrm{op}}, \mathcal{A}b)).$$

Since the category $\mathcal{A}dd(\mathcal{P}^{\mathrm{op}}, \mathcal{A}b)$ is abelian,

$$\mathcal{LH}(\mathcal{A}dd(\mathcal{P}^{\mathrm{op}}, \mathcal{A}b)) \approx \mathcal{A}dd(\mathcal{P}^{\mathrm{op}}, \mathcal{A}b)$$

and the proof is complete. $\square$

Proposition 2.1.15. *Let $\mathcal{E}$ be a quasi-abelian category. Assume $\mathcal{E}$ is quasi-elementary. Then,*

(a) *Both the categories $\mathcal{E}$ and $\mathcal{LH}(\mathcal{E})$ are complete with exact products. Moreover,*

$$I : \mathcal{E} \longrightarrow \mathcal{LH}(\mathcal{E})$$

preserves projective limits.

(b) *Both the categories $\mathcal{E}$ and $\mathcal{LH}(\mathcal{E})$ are cocomplete with strongly exact direct sums. Moreover,*

$$I : \mathcal{E} \longrightarrow \mathcal{LH}(\mathcal{E})$$

preserves direct sums.

(c) *Both the categories $\mathcal{E}$ and $\mathcal{LH}(\mathcal{E})$ have enough projective objects. Moreover, $\mathcal{LH}(\mathcal{E})$ has enough injective objects.*

Proof.
(a) It follows from the preceding proposition that $\mathcal{LH}(\mathcal{E})$ is complete. Hence, Proposition 1.4.8 shows that $\mathcal{E}$ is complete. Thanks to Proposition 1.4.5, $\mathcal{E}$ has exact products. Hence, the conclusion follows from Corollary 1.4.7.
(b) This follows from Proposition 2.1.12.
(c) Obvious. □

Proposition 2.1.16. *Let $\mathcal{E}$ be a quasi-abelian category. Assume $\mathcal{E}$ is quasi-elementary. Then, $\mathcal{E}$ is elementary if and only if one of the following equivalent conditions is satisfied*

(a) *The functor*
$$I : \mathcal{E} \to \mathcal{LH}(\mathcal{E})$$
preserves filtering inductive limits.

(b) *Filtering inductive limits are exact in $\mathcal{E}$.*

(c) *Filtering inductive limits are strongly exact in $\mathcal{E}$.*

Proof. This follows from Proposition 2.1.4, Proposition 2.1.9 and Proposition 1.4.17. □

Proposition 2.1.17. *Let $\mathcal{E}$ be a small quasi-abelian category with enough projective objects. Then,*
$$\mathcal{I}nd(\mathcal{E})$$
is an elementary quasi-abelian category and there is a canonical equivalence of categories
$$\mathcal{LH}(\mathcal{I}nd(\mathcal{E})) \approx \mathcal{I}nd(\mathcal{LH}(\mathcal{E})).$$

Proof. Proceeding as in the abelian case (see [14]), it is easy to check that $\mathcal{I}nd(\mathcal{E})$ is an elementary quasi-abelian category. Denote $\mathcal{P}$ the full additive subcategory of $\mathcal{E}$ formed by projective objects. It follows from Proposition 2.1.14 that
$$\mathcal{LH}(\mathcal{I}nd(\mathcal{E})) \approx \mathcal{A}dd(\mathcal{P}^{\mathrm{op}}, \mathcal{A}b)$$
Since any object of $\mathcal{LH}(\mathcal{E})$ is a quotient of an object of $I(\mathcal{P})$ and since any such object is projective, $\mathcal{I}nd(\mathcal{LH}(\mathcal{E}))$ is an elementary abelian category and
$$\mathcal{I}nd(\mathcal{LH}(\mathcal{E})) \approx \mathcal{A}dd(\mathcal{P}^{\mathrm{op}}, \mathcal{A}b).$$
The conclusion follows easily. □

2.1.4 Closed elementary categories

Proposition 2.1.18. *Let $\mathcal{E}$ be a closed quasi-abelian category with T as internal tensor product and let R be a unital ring in $\mathcal{E}$.*

(a) Assume P is a small (resp. tiny) object of $\mathcal{E}$. Then, $T(R, P)$ is a small (resp. tiny) object of $\mathcal{M}od(R)$.

(b) Assume $\mathcal{G}$ is a strictly generating set of $\mathcal{E}$. Then,

$$\{T(R, G) : G \in \mathcal{G}\}$$

is a strictly generating set of $\mathcal{M}od(R)$.

(c) Assume $\mathcal{E}$ is quasi-elementary (resp. elementary). Then $\mathcal{M}od(R)$ is quasi-elementary (resp. elementary).

Proof. Part (a) follows directly from Proposition 1.5.2.

To prove (b), let E be an R-module and let $s : S \longrightarrow E$ be a monomorphism of $\mathcal{M}od(R)$ which is not an isomorphism. Since $\mathcal{G}$ is a strictly generating set of $\mathcal{E}$, Definition 2.1.5 shows that there is $G \longrightarrow E$ in $\mathcal{E}$ which cannot be factorized through s in $\mathcal{E}$. It follows that the associated morphism $T(R, G) \longrightarrow E$ of $\mathcal{M}od(R)$ cannot be factorized through s in $\mathcal{M}od(R)$ and we get the conclusion.

Part (c) is a direct consequence of (a) and (b). □

Proposition 2.1.19. *Let $\mathcal{E}$ be a small quasi-abelian category with enough projective objects. Assume $\mathcal{E}$ is endowed with a closed structure with T as internal tensor product, H as internal homomorphism functor and U as unit object. Then, there is a closed structure on $\mathcal{I}nd(\mathcal{E})$ which extends that of $\mathcal{E}$ and any two such extensions are canonically isomorphic. Assume moreover that for any projective object P of $\mathcal{E}$ the functor*

$$T(P, \cdot) : \mathcal{E} \longrightarrow \mathcal{E}$$

is exact (resp. strongly exact) and that $T(P, P')$ is projective for any projective object P' of $\mathcal{E}$. Then, similar properties hold for projective objects of $\mathcal{I}nd(\mathcal{E})$.

Proof. Assume $\mathcal{I}nd(\mathcal{E})$ is endowed with a closed structure extending that of $\mathcal{E}$. Denote T' its internal tensor product, H' its internal homomorphism functor and U' its unit object. Using the canonical fully faithful functor

$$\text{“}\cdot\text{”} : \mathcal{E} \longrightarrow \mathcal{I}nd(\mathcal{E})$$

we may express the fact that the closed structure of $\mathcal{I}nd(\mathcal{E})$ extends that of $\mathcal{E}$ by the formulas

$$U' \simeq \text{“}U\text{”}, \quad T'(\text{“}E\text{”}, \text{“}F\text{”}) \simeq \text{“}T(E, F)\text{”}, \quad H'(\text{“}E\text{”}, \text{“}F\text{”}) \simeq \text{“}H(E, F)\text{”}.$$

Let $E : \mathcal{I} \longrightarrow \mathcal{E}$, $F : \mathcal{J} \longrightarrow \mathcal{E}$ and $G : \mathcal{K} \longrightarrow \mathcal{E}$ be three filtering inductive systems of $\mathcal{E}$. It follows from the adjunction formula between T' and H' that

$$T'(\varinjlim_{i\in\mathcal{I}} \text{“}E(i)\text{”}, \varinjlim_{j\in\mathcal{J}} \text{“}F(j)\text{”}) \simeq \varinjlim_{i\in\mathcal{I}} \varinjlim_{j\in\mathcal{J}} \text{“}T(E(i), F(j))\text{”}$$

and that

$$H'(\varinjlim_{j\in\mathcal{J}} \text{“}F(j)\text{”}, \varinjlim_{k\in\mathcal{K}} \text{“}G(k)\text{”}) \simeq \varprojlim_{j\in\mathcal{J}} \varinjlim_{k\in\mathcal{K}} \text{“}H(F(j), G(k))\text{”}.$$

These formulas show directly that T' and H' are unique up to canonical isomorphism. We may also use them to construct a closed structure on $\mathcal{I}nd(\mathcal{E})$ extending that of $\mathcal{E}$ (details are left to the reader).

Since any projective object of $\mathcal{I}nd(\mathcal{E})$ is a direct factor of a projective object of the form

$$\bigoplus_{i\in I} \text{“}P_i\text{”}$$

with P_i projective in $\mathcal{E}$, the last part of the statement is clear. □

2.2 Sheaves with values in an elementary quasi-abelian category

In this section, we will fix a small topological space X and an elementary quasi-abelian category $\mathcal{E}$ and we will show that the category of sheaves with values in $\mathcal{E}$ can be manipulated almost as easily as the category of abelian sheaves.

2.2.1 Presheaves, sheaves and the associated sheaf functor

Definition 2.2.1. A *presheaf* on X with values in $\mathcal{E}$ (or $\mathcal{E}$-*presheaf*) is a functor

$$F : \mathcal{O}p(X)^{\mathrm{op}} \longrightarrow \mathcal{E}$$

where $\mathcal{O}p(X)$ denotes the category of open subsets of X with the inclusion maps as morphisms. If $V \subset U$ are two open subsets of X, we denote

$$r^F_{V,U} : F(U) \longrightarrow F(V)$$

the associated restriction morphism. We define the category of presheaves on X with values in $\mathcal{E}$ by setting

$$\mathcal{P}sh(X; \mathcal{E}) = \mathcal{E}^{\mathcal{O}p(X)^{\mathrm{op}}}.$$

Let F be an object of $\mathcal{P}sh(X;\mathcal{E})$. We define the fiber F_x of F at $x \in X$ by setting

$$F_x = \varinjlim_{V \in \mathcal{V}_x^{\mathrm{op}}} F(V)$$

where $\mathcal{V}_x$ denotes the set of open neighborhoods of x ordered by inclusion.

For any $U \in \mathcal{V}_x$, we denote

$$r^F_{x,U} : F(U) \longrightarrow F_x$$

the canonical morphism.

For any open subset U of X and any $\mathcal{E}$-presheaf F, we denote $F_{|U}$ the $\mathcal{E}$-presheaf obtained by restricting the functor F to $\mathcal{O}p(U)^{\mathrm{op}}$.

An object F of $\mathcal{P}sh(X;\mathcal{E})$ is a *mono-presheaf* if for any open subset U of X and any covering $\mathcal{V}$ of U, the morphism

$$r : F(U) \longrightarrow \prod_{V \in \mathcal{V}} F(V)$$

defined by setting $p_V \circ r = r^F_{V,U}$ is monomorphic. Equivalently, F is a mono-presheaf if and only if $h_E \circ F$ is an abelian mono-presheaf for any object E of $\mathcal{E}$.

An object F of $\mathcal{P}sh(X;\mathcal{E})$ is a *sheaf* (or $\mathcal{E}$*-sheaf*) if for any open subset U of X and any covering $\mathcal{V}$ of U we get the strict exact sequence

$$0 \longrightarrow F(U) \xrightarrow{r} \prod_{V \in \mathcal{V}} F(V) \xrightarrow{r'} \prod_{W,W' \in \mathcal{V}} F(W \cap W')$$

where r is defined as above and

$$p_{W,W'} \circ r' = r^F_{W\cap W',W} \circ p_W - r^F_{W \cap W',W'} \circ p_{W'}.$$

Equivalently, F is a sheaf if and only if $h_E \circ F$ is an abelian for any object E of $\mathcal{E}$.

We denote by $\mathcal{S}hv(X;\mathcal{E})$ the full subcategory of $\mathcal{P}sh(X;\mathcal{E})$ formed by sheaves.

Proposition 2.2.2. *The category $\mathcal{P}sh(X;\mathcal{E})$ is a quasi-abelian category and*

$$\mathcal{LH}(\mathcal{P}sh(X;\mathcal{E})) \simeq \mathcal{P}sh(X;\mathcal{LH}(\mathcal{E})).$$

Moreover, if $\mathcal{P}$ is a strictly generating full additive subcategory of small projective objects of $\mathcal{E}$. Then, the canonical functor

$$h : \mathcal{P}sh(X;\mathcal{E}) \longrightarrow \mathcal{A}dd(\mathcal{P}, \mathcal{P}sh(X;\mathcal{A}b))$$

which associates to an $\mathcal{E}$-presheaf F the functor

$$P \mapsto h_P \circ F$$

factors through an equivalence of categories

$$\mathcal{LH}(\mathcal{P}sh(X;\mathcal{E})) \longrightarrow \mathcal{A}dd(\mathcal{P};\mathcal{P}sh(X;\mathcal{A}b)).$$

In particular, h is a fully faithful strictly exact functor.

Proof. The first part of the result follows from Proposition 1.4.9. Since $\mathcal{E}$ has exact direct sums, it follows from Proposition 1.4.15 that

$$\mathcal{LH}(\mathcal{P}sh(X;\mathcal{E})) \simeq \mathcal{P}sh(X;\mathcal{LH}(\mathcal{E})).$$

Since Proposition 2.1.14 shows that

$$\mathcal{LH}(\mathcal{E}) \simeq \mathcal{A}dd(\mathcal{P};\mathcal{A}b),$$

the conclusion follows easily. □

Definition 2.2.3. Let $\mathcal{V}$ be a covering of X. We define $L(\mathcal{V};F)$ to be the kernel of the morphism

$$\prod_{V\in\mathcal{V}} F(V) \xrightarrow{r'} \prod_{W,W'\in\mathcal{V}} F(W\cap W').$$

We set also

$$L(X;F) = \varinjlim_{\mathcal{V}\in\mathcal{C}v(X)^{\mathrm{op}}} L(\mathcal{V};F)$$

where $\mathcal{C}v(X)$ denotes the set of open coverings of X ordered by setting $\mathcal{V} \leq \mathcal{V}'$ if for any $V \in \mathcal{V}$ there is $V' \in \mathcal{V}'$ such that $V \subset V'$.

Finally, we define the $\mathcal{E}$-presheaf $L(F)$ by setting

$$L(F)(U) = L(U;F_{|U}).$$

We have a canonical morphism

$$F \longrightarrow L(F).$$

Proposition 2.2.4. *(a) For any $\mathcal{E}$-presheaf F, $L(F)$ is an $\mathcal{E}$-mono-presheaf.*

(b) For any $\mathcal{E}$-mono-presheaf F, $L(F)$ is an $\mathcal{E}$-sheaf.

Proof. $L(F)$ is an $\mathcal{E}$-mono-presheaf (resp. an $\mathcal{E}$-sheaf) if and only if $h_P \circ L(F)$ is an abelian mono-presheaf (resp. an abelian sheaf) for any tiny projective object P of $\mathcal{E}$. Since it is clear that

$$h_P \circ L(F) = L(h_P \circ F)$$

we are reduced to the case where $\mathcal{E} = \mathcal{A}b$ which is well-known. □

Definition 2.2.5. We define the associated sheaf functor

$$A : \mathcal{P}sh(X;\mathcal{E}) \longrightarrow \mathcal{S}hv(X;\mathcal{E})$$

by setting $A = L \circ L$. We have a canonical morphism

$$a(F) : F \longrightarrow A(F).$$

Proposition 2.2.6. *For any morphism*

$$u : F \longrightarrow G$$

from the presheaf F to the sheaf G there is a unique morphism

$$v : A(F) \longrightarrow G$$

making the diagram

$$\begin{tikzcd} F \arrow[r, "a(F)"] \arrow[d, "u"'] & A(F) \arrow[dl, "v"] \\ G & \end{tikzcd}$$

commutative. Therefore, $F \simeq A(F)$ if and only if F is a sheaf, and we have the adjunction isomorphism

$$\mathrm{Hom}_{\mathcal{S}hv(X;\mathcal{E})}(A(F), G) \simeq \mathrm{Hom}_{\mathcal{P}sh(X;\mathcal{E})}(F, G)$$

which shows that $\mathcal{S}hv(X;\mathcal{E})$ is a reflective subcategory of $\mathcal{P}sh(X;\mathcal{E})$.

Moreover, for any $x \in X$, we have a canonical isomorphism

$$A(F)_x \simeq F_x.$$

Proof. We have

$$h[A(F)][P] = A[h(F)[P]].$$

Hence, there is a unique morphism

$$v'[P] : A[h(F)[P]] \longrightarrow h(G)[P]$$

such that

$$v'[P] \circ a(h(F)[P]) = h(u)[P].$$

Since h is full there is a morphism

$$v : A(F) \longrightarrow G$$

such that
$$h(v) = v'.$$
For such a v, we get
$$h(v \circ a(F))[P] = v' \circ a(h(F)[P]) = h(u)[P]$$
and since h is faithful, we have
$$v \circ a(F) = u.$$
Moreover, any morphism
$$w : A(F) \longrightarrow G$$
such that
$$w \circ a(F) = u$$
satisfies the equality
$$h(w) = v'$$
and h being faithful, we get
$$w = v.$$
Since
$$h[A(F)][P] = A(h(F)[P]),$$
we get
$$h_P(A(F)_x) = h[A(F)][P]_x \simeq h(F)[P]_x = h_P(F_x)$$
and the last part of the result follows easily. □

2.2.2 The category of sheaves

Proposition 2.2.7. *The category*
$$\mathcal{S}hv(X; \mathcal{E})$$
is quasi-abelian. Moreover, a sequence
$$E \longrightarrow F \longrightarrow G$$
is strictly exact (resp. coexact) in $\mathcal{S}hv(X; \mathcal{E})$ if and only if the sequence
$$E_x \longrightarrow F_x \longrightarrow G_x$$
is strictly exact (resp. coexact) in $\mathcal{E}$ for any $x \in X$. In particular, a morphism
$$u : E \longrightarrow F$$

of $\mathcal{S}hv(X;\mathcal{E})$ is strict if and only if

$$u_x : E_x \longrightarrow F_x$$

is strict for any $x \in X$.

Proof. Let

$$u : E \longrightarrow F$$

be a morphism in $\mathcal{S}hv(X;\mathcal{E})$. Define the object K of $\mathcal{P}sh(X;\mathcal{E})$ by setting

$$K(U) = \mathrm{Ker} u(U)$$

for any open subset U of X. Since

$$h(K) = \mathrm{Ker} h(u),$$

it is clear that K is an $\mathcal{E}$-sheaf. By construction, K is a kernel of u in $\mathcal{S}hv(X;\mathcal{E})$. Define the object C of $\mathcal{P}sh(X;\mathcal{E})$ by setting

$$C(U) = \mathrm{Coker} u(U).$$

Since C is a cokernel of u in $\mathcal{P}sh(X;\mathcal{E})$, the adjunction formula for A shows that $A(C)$ is a cokernel of u in $\mathcal{S}hv(X;\mathcal{E})$.

It follows from what precedes that any morphism

$$u : E \longrightarrow F$$

of $\mathcal{S}hv(X;\mathcal{E})$ has a kernel, a cokernel, an image and a coimage and that

$$(\mathrm{Ker} u)_x \simeq \mathrm{Ker}(u_x), \quad (\mathrm{Im} u)_x \simeq \mathrm{Im}(u_x),$$

$$(\mathrm{Coker} u)_x \simeq \mathrm{Coker}(u_x), \quad (\mathrm{Coim} u)_x \simeq \mathrm{Coim}(u_x).$$

Therefore, to conclude, it is sufficient to prove that u is an isomorphism if u_x is an isomorphism for every $x \in X$. Since

$$h(u)[P]_x \simeq h_P(u_x)$$

this is a direct consequence of the corresponding result for abelian sheaves. □

Proposition 2.2.8. *The category $\mathcal{S}hv(X;\mathcal{E})$ is complete and cocomplete. Moreover, direct sums and filtering inductive limits are strongly exact.*

Proof. Let

$$F : \mathcal{J} \longrightarrow \mathcal{S}hv(X;\mathcal{E})$$

be a functor and let L be the projective limit of F in $\mathcal{P}sh(X;\mathcal{E})$. Since

$$h_X \circ L = \varprojlim_{j \in \mathcal{J}} h_X \circ F(j)$$

for any object X of $\mathcal{E}$, L is an $\mathcal{E}$-sheaf. Hence, L is a projective limit of F in $\mathcal{S}hv(X;\mathcal{E})$. Let R be the inductive limit of F in $\mathcal{P}sh(X;\mathcal{E})$. It follows from the adjunction formula for A that $A(R)$ is an inductive limit of F in $\mathcal{S}hv(X;\mathcal{E})$. The last part follows from the fact that in $\mathcal{S}hv(X;\mathcal{E})$ we have

$$(\varinjlim_{j \in \mathcal{J}} F(j))_x = \varinjlim_{j \in \mathcal{J}} F(j)_x.$$

□

Definition 2.2.9. Let E be an object of $\mathcal{E}$ and let U be an open subset of X. Consider the $\mathcal{E}$-presheaf F defined by setting

$$F(V) = \begin{cases} E & \text{if} \quad V \subset U \\ 0 & \text{if} \quad V \not\subset U \end{cases}$$

the restriction map

$$r^G_{WV} : F(V) \longrightarrow F(W)$$

being id_E if $W \subset V$ and 0 if $W \not\subset V$. By construction, for any $\mathcal{E}$-presheaf G, we have

$$\operatorname{Hom}_{\mathcal{P}sh(X;\mathcal{E})}(F, G) \simeq \operatorname{Hom}_{\mathcal{E}}(E, G(U)).$$

We set

$$E_U = A(F).$$

Clearly,

$$\operatorname{Hom}_{\mathcal{S}hv(X;\mathcal{E})}(E_U, G) \simeq \operatorname{Hom}_{\mathcal{E}}(E, G(U)).$$

Proposition 2.2.10. *For any open subset U of X, the functor*

$$(\cdot)_U : \mathcal{E} \longrightarrow \mathcal{S}hv(X;\mathcal{E})$$

is strictly exact and preserves inductive limits. Moreover

$$(E_U)_x \simeq \begin{cases} E & \text{if} \quad x \in U \\ 0 & \text{if} \quad x \notin U \end{cases}$$

Proposition 2.2.11. *Let $\mathcal{G}$ be a strictly generating small set of objects of $\mathcal{E}$. Then,*

$$\{G_U : G \in \mathcal{G}, \quad U \text{ open subset of } X\}$$

is a strictly generating small set of objects of $\mathcal{S}hv(X;\mathcal{E})$.

Proof. Consider the canonical morphism

$$u : \bigoplus_{U \in \mathcal{O}p(X)} F(U)_U \to F$$

corresponding to the morphism

$$F(U)_U \to F$$

deduced from the identity morphism

$$F(U) \to F(U).$$

For any $x \in X$, we have

$$(\bigoplus_{U \in \mathcal{O}p(X)} F(U)_U)_x \simeq \bigoplus_{x \in U} F(U).$$

The morphism

$$\bigoplus_{U \ni x} F(U) \to F_x$$

induced by u_x corresponds to the restriction morphisms

$$r^F_{x,U} : F(U) \to F_x.$$

Hence, u_x is a strict epimorphism for any $x \in X$. This shows that u is a strict epimorphism. Since $\mathcal{G}$ is a small strictly generating set of $\mathcal{E}$, for any $U \in \mathcal{O}p(X)$, there is a small family $(G_{U,i})_{i \in I_U}$ of $\mathcal{G}$ and a strict epimorphism of $\mathcal{E}$

$$\bigoplus_{i \in I_U} G_{U,i} \to F(U).$$

Since $(\cdot)_U$ preserves inductive limits,

$$\bigoplus_{i \in I_U} (G_{U,i})_U \to F(U)_U$$

is a strict epimorphism in $\mathcal{S}hv(X;\mathcal{E})$. Hence, we get a strict epimorphism

$$\bigoplus_{U \in \mathcal{O}p(X)} \bigoplus_{i \in I_U} (G_{U,i})_U \to F.$$

The conclusion follows easily. $\square$

Proposition 2.2.12. *The canonical inclusion*

$$I : \mathcal{E} \to \mathcal{LH}(\mathcal{E})$$

gives a canonical functor

$$\mathcal{S}hv(X;\mathcal{E}) \to \mathcal{S}hv(X;\mathcal{LH}(\mathcal{E})).$$

This functor induces an equivalence of categories

$$\mathcal{LH}(\mathcal{S}hv(X;\mathcal{E})) \simeq \mathcal{S}hv(X;\mathcal{LH}(\mathcal{E})).$$

Proof. Since I is continuous, $I \circ F$ is an $\mathcal{E}$-sheaf for any $\mathcal{E}$-sheaf F. This gives us a canonical functor

$$J : \mathcal{S}hv(X;\mathcal{E}) \to \mathcal{S}hv(X;\mathcal{LH}(\mathcal{E})).$$

Ones checks easily that J is fully faithful and that its essential image is stable by subobjects. Moreover,

$$J(F)_x \simeq I(F_x).$$

Let $\mathcal{G}$ be a strictly generating small set of objects of $\mathcal{E}$. We know that $I(\mathcal{G})$ is a generating small set of $\mathcal{LH}(\mathcal{E})$. Hence, for any object F of $\mathcal{S}hv(X;\mathcal{LH}(\mathcal{E}))$, there is a small family $(U_l, G_l)_{l\in L}$ of $\mathcal{O}p(X) \times \mathcal{G}$ and a strict epimorphism

$$\bigoplus_{l\in L}(I(G_l))_{U_l} \to F.$$

Moreover, since I commutes with filtering inductive limits, one checks easily that

$$(I(G_l))_{U_l} = J((G_l)_{U_l})$$

and that

$$\bigoplus_{l\in L} J((G_l)_{U_l}) \simeq J(\bigoplus_{l\in L}(G_l)_{U_l}).$$

Hence, we have an epimorphism

$$J(\bigoplus_{l\in L}(G_l)_{U_l}) \to F.$$

The conclusion follows from Proposition 1.2.35. $\square$

2.2.3 Internal operations on sheaves

Let $\mathcal{E}$ be a closed elementary quasi-abelian category and let T (resp. H, U) be its internal tensor product (resp. its internal homomorphism functor, its unit object).

Definition 2.2.13. Let F and G be two objects of $\mathcal{P}sh(X;\mathcal{E})$. We denote by $H(F,G)$ the kernel of the morphism

$$h : \prod_{U\in\mathcal{O}p(X)} H(F(U),G(U)) \to \prod_{\substack{U,V\in\mathcal{O}p(X)\\ V\subset U}} H(F(U),G(V))$$

of $\mathcal{E}$ defined by setting

$$p_{UV}\circ h = H(\mathrm{id}_{F(U)}, r^G_{VU})\circ p_U - H(r^F_{VU}, \mathrm{id}_{G(V)})\circ p_V$$

for any $V\subset U$ in $\mathcal{O}p(X)$. Clearly,

$$U\mapsto H(F|_U, G|_U)$$

is a presheaf. We denote it by $\mathcal{H}(F,G)$.

Proposition 2.2.14. *Assume F, G are two objects of $\mathcal{S}hv(X;\mathcal{E})$. Then, $\mathcal{H}(F,G)$ is an object of $\mathcal{S}hv(X;\mathcal{E})$.*

Proof. Let $V\subset U$ be open subsets of X and let $\mathcal{W}$ be a covering of X. Since G is a sheaf, we have the strictly exact sequence

$$0\to G(V)\to \prod_{W\in\mathcal{W}} G(V\cap W)\to \prod_{W,W'\in\mathcal{W}} G(V\cap W\cap W').$$

From the adjunction formula linking T and H, it follows that $H(F(U),\cdot)$ is a continuous functor. Therefore, we get the strictly exact sequence

$$0\to H(F(U),G(V))\to \prod_{W\in\mathcal{W}} H(F(U),G(V\cap W))\to \prod_{W,W'\in\mathcal{W}} H(F(U),G(V\cap W\cap W')).$$

Using a tedious but easy computation, we deduce from this fact that the sequence

$$0\to \mathcal{H}(F,G)(X)\to \prod_{W\in\mathcal{W}} \mathcal{H}(F,G)(W)\to \prod_{W,W'\in\mathcal{W}} \mathcal{H}(F,G)(W\cap W')$$

is strictly exact in $\mathcal{E}$. And the conclusion follows. $\square$

Definition 2.2.15. Let E, F be two objects of $\mathcal{S}hv(X;\mathcal{E})$. We denote $\mathcal{T}(E,F)$ the $\mathcal{E}$-sheaf associated to the $\mathcal{E}$-presheaf

$$U\mapsto T(E(U),F(U)).$$

Proposition 2.2.16. *We have the canonical functorial isomorphism*

$$\operatorname{Hom}_{\mathcal{S}hv(X;\mathcal{E})}(\mathcal{T}(E,F),G) \simeq \operatorname{Hom}_{\mathcal{S}hv(X;\mathcal{E})}(E,\mathcal{H}(F,G))$$

for E, F, G in $\mathcal{S}hv(X;\mathcal{E})$.

Proof. Let $h \in \operatorname{Hom}_{\mathcal{S}hv(X;\mathcal{E})}(\mathcal{T}(E,F),G)$ and let $U \supset V$ be open subsets of X. By composition with the canonical morphism

$$T(E(U),F(V)) \to T(E(V),F(V)) \to \mathcal{T}(E,F)(V),$$

the morphism

$$h(V) : \mathcal{T}(E,F)(V) \to G(V)$$

induces a morphism

$$h'_{UV} : T(E(U),F(V)) \to G(V).$$

By adjunction, this gives us a morphism

$$h''_{UV} : E(U) \to H(F(V),G(V))$$

for any $V \subset U$ in $\mathcal{O}p(X)$. Hence, we get a morphism

$$h''_U : E(U) \to \prod_{\substack{V \subset U \\ V \text{ open}}} H(F(V),G(V))$$

and one checks easily that h''_U factors through

$$h'''_U : E(U) \to H(F|_U, G|_U).$$

Moreover, the family $(h'''_U)_{U \in \mathcal{O}p(X)}$ defines a morphism

$$\varphi(h) : E \to \mathcal{H}(F,G)$$

in $\mathcal{S}hv(X;\mathcal{E})$.

Now, let $h \in \operatorname{Hom}_{\mathcal{S}hv(X;\mathcal{E})}(E,\mathcal{H}(F,G))$ and let U be an open subset of X. The morphism

$$h(U) : E(U) \to H(F|_U, G|_U)$$

gives rise to a morphism

$$h'(U) : E(U) \to H(F(U),G(U)).$$

By adjunction, we get a morphism

$$h''(U) : T(E(U),F(U)) \to G(U)$$

which is easily checked to be a morphism of presheaves. We denote

$$\psi(h) : \mathcal{T}(E, F) \longrightarrow G$$

the associated morphism of $\mathcal{Shv}(X;\mathcal{E})$. An easy computation shows that ψ is an inverse of φ and that ψ and φ are functorial in E, F and G. $\square$

Corollary 2.2.17. *The category $\mathcal{Shv}(X;\mathcal{E})$ endowed with $\mathcal{T}$ as internal tensor product, U_X as unit and $\mathcal{H}$ as internal homomorphism functor is a closed quasi-abelian category.*

2.3 Sheaves with values in an elementary abelian category

2.3.1 Poincaré-Verdier duality

Let $f : X \longrightarrow Y$ be a continuous map of locally compact topological spaces and let $\mathcal{A}$ be an elementary abelian category. Recall that

$$\mathcal{Shv}(X;\mathcal{A}) \quad (\text{resp. } \mathcal{Shv}(Y;\mathcal{A}))$$

denotes the category of sheaves on X (resp. Y) with values in $\mathcal{A}$. For short, we set

$$\mathcal{D}^*(X,\mathcal{A}) = \mathcal{D}^*(\mathcal{Shv}(X,\mathcal{A})) \quad (* = +, -, \emptyset)$$

and use similar conventions for $\mathcal{K}$.

As usual, for any closed subspace Q of X and any $\mathcal{A}$-sheaf F on X, $\Gamma_Q(X;F)$ denotes the kernel of the restriction morphism

$$F(X) \longrightarrow F(X \setminus Q).$$

Definition 2.3.1. For any sheaf $F \in \mathcal{Shv}(X;\mathcal{A})$, we define the sheaf

$$f_!(F) \in \mathcal{Shv}(Y;\mathcal{A})$$

by the formula

$$\Gamma(U; f_!(F)) = \varinjlim_{Q \subset f^{-1}(U), Q\, f\text{-proper}} \Gamma_Q(f^{-1}(U); F)$$

We call *f-soft* a sheaf F such that F_U is $f_!$-acyclic for any $U \in \mathcal{Op}(X)$.

Of course, $f_!$ is left exact and gives rise to a derived functor

$$Rf_! : \mathcal{D}^+(X;\mathcal{A}) \longrightarrow \mathcal{D}^+(Y;\mathcal{A}).$$

Hereafter, we will show that under the assumption that $f_!$ has finite cohomological dimension, $Rf_!$ has a right adjoint functor. To get this result, we will adapt the reasoning of [6] to our more general situation.

Definition 2.3.2. Let $K(F)$ denote a bounded functorial f-soft resolution of $F \in \mathcal{S}hv(X;\mathcal{A})$ (e.g. a truncated Godement resolution). For any $G \in \mathcal{S}hv(Y;\mathcal{A})$, we define the presheaf

$$f^!_K(G) \in \mathcal{P}sh(X;\mathcal{A})$$

through Proposition 2.2.2 by asking that

$$\mathrm{Hom}\,(P, \Gamma(U; f^!_K(G))) = \mathrm{Hom}\,(f_!(K(P_X)_U), G)$$

for any P in a generating small set of small projective objects of $\mathcal{A}$.

Proposition 2.3.3. *For any $G \in \mathcal{S}hv(Y;\mathcal{A})$, $f^!_K(G) \in \mathcal{S}hv(X;\mathcal{A})$. Moreover, if G is injective, $f^!_K(G)$ is flabby.*

Proof. Let $(U_i)_{i\in I}$ be a covering of an open subset U of X. We know that, for any $k \in \mathbb{Z}$, the complex

$$\cdots \longrightarrow \bigoplus_{i,j\in I} K^k(P_X)_{U_i\cap U_j} \longrightarrow \bigoplus_{i\in I} K^k(P_X)_{U_i} \longrightarrow K^k(P_X)_U \longrightarrow 0$$

is exact. Since $K^k(P_X)_V$ is $f_!$-acyclic for any open subset V of X and $f_!$ has finite cohomological dimension, the complex

$$\cdots \longrightarrow \bigoplus_{i,j\in I} f_!(K^k(P_X)_{U_i\cap U_j}) \longrightarrow \bigoplus_{i\in I} f_!(K^k(P_X)_{U_i}) \longrightarrow f_!(K^k(P_X)_U) \longrightarrow 0$$

is also exact. It follows that the sequence

$$0 \longrightarrow \mathrm{Hom}\,(f_!(K(P_X)_U), G) \longrightarrow \prod_{i\in I} \mathrm{Hom}\,(f_!(K(P_X)_{U_i}), G) \longrightarrow \prod_{i,j\in I} \mathrm{Hom}\,(f_!(K(P_X)_{U_i\cap U_j}), G)$$

is exact. Hence, we see that the sequence

$$0 \longrightarrow \mathrm{Hom}\,(P, \Gamma(U; f^!_K(G))) \longrightarrow \mathrm{Hom}\,(P, \prod_{i\in I} \Gamma(U_i, f^!_K(G))) \longrightarrow \mathrm{Hom}\,(P, \prod_{i,j\in I} \Gamma(U_i \cap U_j; f^!_K(G)))$$

is exact for any P in a small generating family of small projective objects. It follows that the sequence

$$0 \longrightarrow \Gamma(U; f^!_K(G)) \longrightarrow \prod_{i\in I} \Gamma(U_i; f^!_K(G)) \longrightarrow \prod_{i,j\in I} \Gamma(U_i \cap U_j; f^!_K(G))$$

is exact and that $f^!_K(G)$ is a sheaf. Let us assume now that G is injective. Let V be an open subset of U. We have the monomorphism

$$K(P_X)_V \longrightarrow K(P_X)_U.$$

Hence, we get the epimorphism

$$\mathrm{Hom}\,(f_!(K(P_X)_U), G) \longrightarrow \mathrm{Hom}\,(f_!(K(P_X)_V), G).$$

As above, we deduce that

$$\Gamma(U; f^!_K(G)) \longrightarrow \Gamma(V; f^!_K(G))$$

is an epimorphism. $\square$

Definition 2.3.4. Let $\mathcal{I}^+(Y, \mathcal{A})$ denote the full subcategory of $\mathcal{K}^+(Y, \mathcal{A})$ formed by complexes of injective sheaves. We denote by

$$f^! : \mathcal{D}^+(Y; \mathcal{A}) \longrightarrow \mathcal{D}^+(X; \mathcal{A})$$

the functor induced by

$$f^!_K : \mathcal{I}^+(Y; \mathcal{A}) \longrightarrow \mathcal{K}^+(X; \mathcal{A})$$

through the equivalence of categories

$$\mathcal{I}^+(Y; \mathcal{A}) \xrightarrow{\sim} \mathcal{D}^+(Y; \mathcal{A}).$$

Proposition 2.3.5. *There is a canonical morphism of functors*

$$Rf_! f^!(G) \longrightarrow G.$$

Proof. Let G be an injective object of $\mathcal{S}hv(Y; \mathcal{A})$. Since $f^!_K(G)$ is flabby, it is also f-soft and

$$Rf_! f^!(G) \simeq f_! f^!_K(G).$$

By definition,

$$\Gamma(U; f_! f^!_K(G)) = \varinjlim_{Q \subset f^{-1}(U), Q\ f\text{-proper}} \Gamma_Q(f^{-1}(U); f^!_K(G)).$$

From the exact sequence

$$0 \longrightarrow \Gamma_Q(f^{-1}(U); f^!_K(G)) \longrightarrow \Gamma(f^{-1}(U); f^!_K(G)) \longrightarrow \Gamma(f^{-1}(U) \setminus Q; f^!_K(G))$$

we deduce that $\mathrm{Hom}\,(P, \Gamma_Q(f^{-1}(U); f^!_K(G)))$ is a kernel of

$$\mathrm{Hom}\,(f_!(K(P_X)_{f^{-1}(U)}), G) \longrightarrow \mathrm{Hom}\,(f_!(K(P_X)_{f^{-1}(U) \setminus Q}), G)$$

Since the sequence

$$0 \longrightarrow K(P_X)_{f^{-1}(U)\setminus Q} \longrightarrow K(P_X)_{f^{-1}(U)} \longrightarrow K(P_X)_Q \longrightarrow 0$$

is exact, it follows that

$$\mathrm{Hom}\,(P, \Gamma_Q(f^{-1}(U); f_K^!(G))) \simeq \mathrm{Hom}\,(f_!(K(P_X)_Q), G).$$

Since Q is f-proper, we have an obvious map

$$P_U \longrightarrow f_!(P_Q) \longrightarrow f_!(K(P_X)_Q).$$

Hence, there is a canonical morphism

$$\mathrm{Hom}\,(P, \Gamma_Q(f^{-1}(U); f_K^!(G))) \longrightarrow \mathrm{Hom}\,(P, \Gamma(U; G))$$

which gives rise to a canonical morphism

$$\Gamma(U; f_! f_K^!(G)) \longrightarrow \Gamma(U; G)$$

as requested. □

Theorem 2.3.6. *The canonical morphism*

$$\mathrm{RHom}\,(F, f^!(G)) \longrightarrow \mathrm{RHom}\,(Rf_!(F), G)$$

induced by the morphism

$$Rf_! f^!(G) \longrightarrow G$$

is an isomorphism.

Proof. We know that any sheaf $F \in \mathcal{S}hv(X; \mathcal{A})$ has a resolution by sheaves of the type

$$\bigoplus (P_i)_{V_i}$$

where V is an open subset of X and P a member of a generating small set of small projective objects of $\mathcal{A}$. Since f has finite cohomological dimension, we may reduce ourselves to the case where $F = P_U$ and G is injective. In such a case, since $f_K^!(G)$ is flabby, we have

$$\begin{aligned} \mathrm{RHom}\,(P_U, f^!(G)) &\simeq \mathrm{Hom}\,(P, \Gamma(U, f_K^!(G))) \\ &\simeq \mathrm{Hom}\,(f_!(K(P)_U), G) \\ &\simeq \mathrm{RHom}\,(Rf_!(P_U), G) \end{aligned}$$

and the conclusion follows. □

2.3.2 Internal projection formula

In this section, $\mathcal{A}$ denotes a closed elementary abelian category with T as internal tensor product, U as unit object and H as internal homomorphism functor. We assume moreover that for any projective object P of $\mathcal{A}$, $T(P,\cdot)$ and $H(P,\cdot)$ are exact functors. It follows from the results in the previous section that $\mathcal{S}hv(X;\mathcal{A})$ endowed with $\mathcal{T}$ as internal tensor product, $\mathcal{H}$ as internal homomorphism functor and U_X as unit object is a closed abelian category.

Definition 2.3.7. We say that an object P of $\mathcal{S}hv(X;\mathcal{A})$ has projective fibers if P_x is a projective object of $\mathcal{A}$ for any $x \in X$.

Lemma 2.3.8. *(a) Assume P is an $\mathcal{A}$-sheaf with projective fibers. Then,*

$$\mathcal{T}(P,\cdot) : \mathcal{S}hv(X;\mathcal{A}) \longrightarrow \mathcal{S}hv(X;\mathcal{A})$$

is exact. Moreover, if P' is another $\mathcal{A}$-sheaf with projective fibers, then $\mathcal{T}(P,P')$ has projective fibers.

(b) Assume I is an injective $\mathcal{A}$-sheaf. Then,

$$\mathcal{H}(\cdot,I) : \mathcal{S}hv(X;\mathcal{A})^{\mathrm{op}} \longrightarrow \mathcal{S}hv(X;\mathcal{A})$$

is exact. Moreover, if P is an $\mathcal{A}$-sheaf with projective fibers, then $\mathcal{H}(P,I)$ is an injective $\mathcal{A}$-sheaf.

Proof. Part (a) follows directly from the fact that

$$\mathcal{T}(E,F)_x = T(E_x,F_x).$$

To prove the first part of (b), let

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

be an exact sequence of $\mathcal{S}hv(X;\mathcal{A})$. Let P be a projective object of $\mathcal{A}$ and let U be an open subset of X. Since the $\mathcal{A}$-sheaf P_U has projective fibers, it follows that the sequence

$$0 \longrightarrow \mathcal{T}(P_U,E') \longrightarrow \mathcal{T}(P_U,E) \longrightarrow \mathcal{T}(P_U,E'') \longrightarrow 0$$

is exact. Using the fact that I is injective, we get the exact sequence

$$0 \longrightarrow \mathrm{Hom}\,(\mathcal{T}(P_U,E''),I) \longrightarrow \mathrm{Hom}\,(\mathcal{T}(P_U,E),I) \longrightarrow \mathrm{Hom}\,(\mathcal{T}(P_U,E'),I) \longrightarrow 0.$$

The adjunction formula between $\mathcal{T}$ and $\mathcal{H}$ then gives the exact sequence

$$0 \longrightarrow \mathrm{Hom}\,(P_U,\mathcal{H}(E'',I)) \longrightarrow \mathrm{Hom}\,(P_U,\mathcal{H}(E,I)) \longrightarrow \mathrm{Hom}\,(P_U,\mathcal{H}(E',I)) \longrightarrow 0.$$

Since for any $\mathcal{A}$-sheaf F, $\operatorname{Hom}(P_U, F) \simeq \operatorname{Hom}(P, F(U))$, we see that the sequence

$$0 \longrightarrow \mathcal{H}(E', I)(U) \longrightarrow \mathcal{H}(E, I)(U) \longrightarrow \mathcal{H}(E'', I)(U) \longrightarrow 0$$

is exact for any open subset U of X and the conclusion follows.

The last part of (b) is obtained by similar methods. $\square$

Remark 2.3.9. One can also prove that if I is an injective $\mathcal{A}$-sheaf, then $\mathcal{H}(E, I)$ is flabby for any $\mathcal{A}$-sheaf E.

Proposition 2.3.10. *The functor*

$$\mathcal{T} : \mathcal{S}hv(X; \mathcal{A}) \times \mathcal{S}hv(X; \mathcal{A}) \longrightarrow \mathcal{S}hv(X; \mathcal{A})$$

is explicitly left derivable and the functor

$$\mathcal{H} : \mathcal{S}hv(X; \mathcal{A})^{\mathrm{op}} \times \mathcal{S}hv(X; \mathcal{A}) \longrightarrow \mathcal{S}hv(X; \mathcal{A})$$

is explicitly right derivable. Moreover, we have the canonical functorial isomorphisms :

(a) $\mathrm{L}\mathcal{T}(E, F) \simeq \mathrm{L}\mathcal{T}(F, E)$,

(b) $\mathrm{L}\mathcal{T}(U_X, E) \simeq E$,

(c) $\operatorname{RHom}(\mathrm{L}\mathcal{T}(E, F), G) \simeq \operatorname{RHom}(E, \mathrm{R}\mathcal{H}(F, G))$,

(d) $\mathrm{R}\mathcal{H}(U_X, E) \simeq E$.

Proof. Let $\mathcal{P}$ denote the full subcategory of $\mathcal{S}hv(X; \mathcal{A})$ formed by $\mathcal{A}$-sheaves with projective fibers. By Proposition 2.2.11, any $\mathcal{A}$-sheaf F is a quotient of an object of the form

$$\bigoplus_{i \in I} (P_i)_{U_i}$$

where P_i is a projective object of $\mathcal{A}$ and U_i is an open subset of X. Since

$$\left(\bigoplus_{i \in I} (P_i)_{U_i} \right)_x = \bigoplus_{i \in I, U_i \ni x} P_i,$$

it is clear that $\bigoplus_{i \in I} (P_i)_{U_i}$ belongs to $\mathcal{P}$. Hence, any $\mathcal{A}$-sheaf is a quotient of an object of $\mathcal{P}$. By the preceding lemma, it follows that $(\mathcal{S}hv(X; \mathcal{A}), \mathcal{P})$ is $\mathcal{T}$-projective. Hence, $\mathcal{T}$ is explicitly left derivable.

Denote $\mathcal{I}$ the full subcategory of $\mathcal{S}hv(X; \mathcal{A})$ formed by injective $\mathcal{A}$-sheaves. We know already that any object of $\mathcal{S}hv(X, \mathcal{A})$ is a subobject of an object of $\mathcal{I}$. By the preceding lemma, $(\mathcal{S}hv(X; \mathcal{A}), \mathcal{I})$ is $\mathcal{H}$-injective. Hence, $\mathcal{H}$ is explicitly right derivable. The last part of the proposition follows directly by replacing the various objects by suitable resolutions. $\square$

Lemma 2.3.11. *Let X be a locally compact topological space. Let P be a projective object of $\mathcal{A}$ and let E be an $\mathcal{A}$-sheaf on X. Then, the canonical morphism*

$$T(\Gamma_c(X;E),P) \to \Gamma_c(X;\mathcal{T}(E,P_X))$$

is an isomorphism. In particular, $\mathcal{T}(E,P_X)$ is c-soft if E is c-soft.

Proof. We will work as in [6, Lemma 2.5.12].

Without losing any generality, we may assume X is compact. Let $(K_i)_{i\in I}$ be a finite covering of X by compact subsets. Since E is an $\mathcal{A}$-sheaf, we have an exact sequence of the form

$$0 \to \Gamma(X;E) \to \bigoplus_{i\in I} \Gamma(K_i;E) \to \bigoplus_{i,j\in I} \Gamma(K_i\cap K_j;E).$$

The object P being projective in $\mathcal{A}$, the functor $T(\cdot,P)$ is exact and we get the morphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & T(\Gamma(X;E),P) \arrow[r, "\lambda"] \arrow[d, "\alpha"] & \bigoplus_{i\in I} T(\Gamma(K_i;E),P) \arrow[r, "\mu"] \arrow[d, "\beta"] & \bigoplus_{i,j\in I} T(\Gamma(K_i\cap K_j;E),P) \arrow[d, "\gamma"] \\
0 \arrow[r] & \Gamma(X;\mathcal{T}(E,P_X)) \arrow[r, "\lambda'"'] & \bigoplus_{i\in I} \Gamma(K_i;\mathcal{T}(E,P_X)) \arrow[r, "\mu'"'] & \bigoplus_{i,j\in I} \Gamma(K_i\cap K_j;\mathcal{T}(E,P_X))
\end{tikzcd}$$

Let us show that α is a monomorphism. It is sufficient to show that for any small projective object Q of $\mathcal{A}$ and any

$$h : Q \to T(\Gamma(X;E),P)$$

such that $\alpha\circ h = 0$ we have $h = 0$. Since

$$\varinjlim_{\substack{U\ni x\\ U \text{ open}}} T(\Gamma(U;E),P) \simeq T(E_x,P) \simeq \varinjlim_{\substack{U\ni x\\ U \text{ open}}} \Gamma(U;\mathcal{T}(E,P_X)) \tag{*}$$

and Q is tiny (see Remark 2.1.2), we can find a finite compact covering of X such that

$$\lambda\circ h = 0.$$

Since λ is monomorphic, the conclusion follows.

To show that α is epimorphic, it is sufficient to show that for any small projective object Q of $\mathcal{A}$ and any

$$h : Q \to \Gamma(X;\mathcal{T}(E,P_X))$$

there is

$$h' : Q \longrightarrow T(\Gamma(X;E),P)$$

such that $\alpha \circ h' = h$. Using once more (*) and the fact that Q is tiny, we can find a finite compact covering of X such that

$$\lambda' \circ h = \beta \circ h''$$

for some

$$h'' : Q \longrightarrow \bigoplus_{i\in I} T(\Gamma(K_i;E),P).$$

It follows from the first part of the proof that β and γ are monomorphic. Since

$$0 = \mu' \circ \lambda' \circ h = \mu' \circ \beta \circ h'' = \gamma \circ \mu \circ h''$$

we see that $\mu \circ h'' = 0$. Hence,

$$h'' = \lambda \circ h'$$

for some

$$h' : Q \longrightarrow T(\Gamma(X;E),P).$$

For such an h', we have

$$\lambda' \circ \alpha \circ h' = \beta \circ \lambda \circ h' = \beta \circ h'' = \lambda' \circ h.$$

Hence, $\alpha \circ h' = h$ and the proof is complete. □

Lemma 2.3.12. *Let $f : X \longrightarrow Y$ be a morphism of locally compact topological spaces. Let P be an $\mathcal{A}$-sheaf on Y with projective fibers and let E be an $\mathcal{A}$-sheaf on X. Then, the canonical morphism*

$$\mathcal{T}(f_!E,P) \longrightarrow f_!\mathcal{T}(E,f^{-1}P)$$

is an isomorphism. Moreover, $\mathcal{T}(E,f^{-1}P)$ is f-soft if E is f-soft.

Proof. Since

$$\mathcal{T}(f_!E,P)_y = T((f_!E)_y,P_y) = T(\Gamma_c(f^{-1}(y);E),P_y)$$

and

$$f_!(\mathcal{T}(E,f^{-1}P))_y = \Gamma_c(f^{-1}(y);\mathcal{T}(E|_{f^{-1}(y)},(P_y)|_{f^{-1}(y)}))$$

for any $y \in Y$, we are reduced to the preceding lemma. □

Proposition 2.3.13. *Let $f : X \longrightarrow Y$ be a morphism of locally compact topological spaces. Assume $f_!$ has finite cohomological dimension. Then,*

$$\mathrm{L}\mathcal{T}(Rf_!E, F) \simeq Rf_!\mathrm{L}\mathcal{T}(E, f^{-1}F)$$

for any E in $\mathcal{D}^-(\mathcal{S}hv(X;\mathcal{A}))$ and any F in $\mathcal{D}^-(\mathcal{S}hv(Y;\mathcal{A}))$.

Proof. This follows directly from the preceding lemma if we replace E by a soft resolution and F by a resolution by $\mathcal{A}$-sheaves with projective fibers. □

2.3.3 Internal Poincaré-Verdier duality

Proposition 2.3.14. *Let $f : X \longrightarrow Y$ be a morphism of locally compact topological spaces. Assume $f_!$ has finite cohomological dimension. Then, the canonical morphism*

$$Rf_*\mathrm{R}\mathcal{H}(E, f^!F) \to \mathrm{R}\mathcal{H}(Rf_!E, F)$$

induced by

$$Rf_!f^!F \longrightarrow F$$

is an isomorphism in $\mathcal{D}^+(\mathcal{S}hv(Y;\mathcal{A}))$ for any E in $\mathcal{D}^-(\mathcal{S}hv(X;\mathcal{A}))$ and any F in $\mathcal{D}^+(\mathcal{S}hv(Y;\mathcal{A}))$.

Proof. It is sufficient to prove that

$$\mathrm{R}\Gamma(f^{-1}(V);\mathrm{R}\mathcal{H}(E, f^!F)) \longrightarrow \mathrm{R}\Gamma(V;\mathrm{R}\mathcal{H}(Rf_!E, F))$$

is an isomorphism for any open subset V of Y. We may even restrict ourselves to the case $V = Y$ and prove only that

$$\mathrm{RHom}\,(P, \mathrm{R}\Gamma(Y;\mathrm{R}\mathcal{H}(E, f^!F))) \xrightarrow{\sim} \mathrm{RHom}\,(P, \mathrm{R}\Gamma(X;\mathrm{R}\mathcal{H}(Rf_!E, F)))$$

for any projective object P of $\mathcal{A}$. This follows from the chain of isomorphism below:

$$\begin{aligned}
\mathrm{RHom}\,(P, \mathrm{R}\Gamma(Y;\mathrm{R}\mathcal{H}(E, f^!F))) &\simeq \mathrm{RHom}\,(P_Y, \mathrm{R}\mathcal{H}(E, f^!F)) \\
&\simeq \mathrm{RHom}\,(\mathrm{L}\mathcal{T}(E, P_Y), f^!F) \\
&\simeq \mathrm{RHom}\,(Rf_!\mathrm{L}\mathcal{T}(E, f^{-1}P_X), F) \\
&\simeq \mathrm{RHom}\,(\mathrm{L}\mathcal{T}(Rf_!E, P_X), F) \\
&\simeq \mathrm{RHom}\,(P_X, \mathrm{R}\mathcal{H}(Rf_!E, F)) \\
&\simeq \mathrm{RHom}\,(P, \mathrm{R}\Gamma(X;\mathrm{R}\mathcal{H}(Rf_!E, F))).
\end{aligned}$$

□

Chapter 3

Applications

3.1 Filtered Sheaves

3.1.1 The category of filtered abelian groups

To fix the notations, let us recall the following definitions.

Definition 3.1.1. A *filtration* on an abelian group M is the data of an increasing sequence $(F_k)_{k\in\mathbb{Z}}$ of abelian subgroups of M such that

$$\bigcup_{k\in\mathbb{Z}} F_k = M.$$

A *filtered abelian group* M is an abelian group M_∞ endowed with a filtration

$$(M_k)_{k\in\mathbb{Z}}.$$

We call M_∞ the underlying abelian group of M.

A *morphism of filtered abelian groups* $u : M \longrightarrow N$ is the data of a morphism

$$u_\infty : M_\infty \longrightarrow N_\infty$$

of the underlying abelian groups such that

$$u_\infty(M_k) \subset N_k$$

for any $k \in \mathbb{Z}$. The set of morphisms from M to N is denoted

$$\mathrm{Hom}\,(M, N).$$

It is clearly endowed with a canonical structure of abelian groups. With this notion of morphisms, one checks easily that filtered abelian groups form an additive category. We will denote it by $\mathcal{F}\mathcal{A}b$.

The following two obvious propositions will clarify the structure of limits in $\mathcal{F}\mathcal{A}b$.

Proposition 3.1.2. *The category $\mathcal{F}\mathcal{A}b$ has kernels and cokernels. More precisely, let $u : M \longrightarrow N$ be a morphism of filtered abelian groups. Then,*

(a) $\operatorname{Ker} u$ *is the abelian group $u_\infty^{-1}(0)$ endowed with the filtration*

$$(u_\infty^{-1}(0) \cap M_k)_{k\in\mathbb{Z}},$$

(b) $\operatorname{Coker} u$ *is the abelian group $N_\infty/u_\infty(M_\infty)$ endowed with the filtration*

$$(N_k + u_\infty(M_\infty)/u_\infty(M_\infty))_{k\in\mathbb{Z}}.$$

As a consequence, we see that

(c) $\operatorname{Im} u$ *is the abelian group $u_\infty(M_\infty)$ endowed with the filtration*

$$(u_\infty(M_\infty) \cap N_k)_{k\in\mathbb{Z}},$$

(d) $\operatorname{Coim} u$ *is the abelian group $M_\infty/u_\infty^{-1}(0)$ endowed with the filtration*

$$(M_k + u_\infty^{-1}(0)/u_\infty^{-1}(0))_{k\in\mathbb{Z}}.$$

It may equivalently be described as the group $u_\infty(M_\infty)$ endowed with the filtration

$$(u_\infty(M_k))_{k\in\mathbb{Z}}.$$

In particular, the morphism u is strict if and only if

$$u_\infty(M_k) = u_\infty(M_\infty) \cap N_k$$

for every $k \in \mathbb{Z}$.

Proposition 3.1.3. *The category $\mathcal{F}\mathcal{A}b$ has direct sums and products. More precisely, let $(M_i)_{i\in I}$ be a small family of filtered abelian groups. Then,*

(a) $\bigoplus_{i\in I} M_i$ is the abelian group $\bigoplus_{i\in I}(M_i)_\infty$ endowed with the filtration

$$\left(\bigoplus_{i\in I}(M_i)_k\right)_{k\in\mathbb{Z}},$$

(b) $\prod_{i\in I} M_i$ is the abelian subgroup

$$\bigcup_{k\in\mathbb{Z}}\prod_{i\in I}(M_i)_k$$

of $\prod_{i\in I}(M_i)_\infty$ endowed with the filtration

$$\left(\prod_{i\in I}(M_i)_k\right)_{k\in\mathbb{Z}}.$$

Remark 3.1.4. It follows from the last point of Proposition 3.1.2 that $\mathcal{F}\mathcal{A}b$ is <u>not</u> abelian. Moreover, Proposition 3.1.3 shows that if I is infinite, $(\prod_{i\in I} M_i)_\infty$ may differ from $\prod_{i\in I}(M_i)_\infty$.

Proposition 3.1.5. *The category $\mathcal{F}\mathcal{A}b$ is a complete and cocomplete quasi-abelian category in which direct sums and filtering inductive limits (resp. products) are strongly exact (resp. exact).*

Proof. It is direct consequence of the two preceding propositions. □

Definition 3.1.6. Let M be a filtered abelian group and let $l \in \mathbb{Z}$. We denote by $M(l)$ the filtered abelian group obtained by endowing M_∞ with the filtration

$$(M_{l+k})_{k\in\mathbb{Z}}.$$

Clearly,

$$M \mapsto M(l)$$

is a functor of $\mathcal{F}\mathcal{A}b$ into itself. We call it the *filtration shifting* functor. Let M, N be two filtered abelian groups.

The sequence

$$(\mathrm{Hom}\,(M, N(k)))_{k\in\mathbb{Z}}$$

of subgroups of

$$\mathrm{Hom}\,(M, N)$$

is increasing and gives a filtration of

$$\bigcup_{k\in\mathbb{Z}} \mathrm{Hom}\,(M, N(k)).$$

We denote $\mathrm{FHom}\,(M, N)$ the corresponding filtered abelian group.

Denote $(M \otimes N)_k$ the image of the canonical morphism

$$\bigoplus_{l\in\mathbb{Z}} M_l \otimes N_{k-l} \longrightarrow M_\infty \otimes N_\infty$$

induced by the canonical inclusions

$$M_l \longrightarrow M_\infty, \qquad N_{k-l} \longrightarrow N_\infty.$$

Clearly, $((M \otimes N)_k)_{k\in\mathbb{Z}}$ forms a filtration of $M_\infty \otimes N_\infty$. We denote by $M \otimes N$ the corresponding filtered abelian group.

Finally, we denote by $\mathrm{F}\mathbb{Z}$ the filtered abelian group obtained by endowing $\mathbb{Z}$ with the filtration defined by setting

$$\mathrm{F}\mathbb{Z}_k = \begin{cases} \mathbb{Z} & \text{if } k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.1.7. *The category $\mathcal{F}\mathcal{A}b$ endowed with $\cdot \otimes \cdot$ as internal tensor product, $\mathrm{FHom}\,(\cdot,\cdot)$ as internal Hom-functor and $\mathrm{F}\mathbb{Z}$ as internal unit forms a closed additive category. In particular, we have*

(a) $\mathrm{Hom}\,(M \otimes N, P) \simeq \mathrm{Hom}\,(M, \mathrm{FHom}\,(N, P))$,

(b) $M \otimes N \simeq N \otimes M$,

(c) $M \otimes \mathrm{F}\mathbb{Z} \simeq M$,

for any objects M, N, P of $\mathcal{F}\mathcal{A}b$.

Proposition 3.1.8. *(a) For any $l \in \mathbb{Z}$, we have*

$$\mathrm{FHom}\,(\mathrm{F}\mathbb{Z}(-l), M) \simeq M_l$$

for any object M of $\mathcal{F}\mathcal{A}b$. In particular, $\mathrm{F}\mathbb{Z}(-l)$ is a tiny projective object of $\mathcal{F}\mathcal{A}b$.

(b) For any object M of $\mathcal{F}\mathcal{A}b$, the canonical morphism

$$\bigoplus_{l\in\mathbb{Z}} \bigoplus_{h \in M_l} \mathrm{F}\mathbb{Z}(-l) \to M$$

induced by the preceding isomorphism is a strict epimorphism. In particular,

$$(\mathrm{F}\mathbb{Z}(-l))_{l\in\mathbb{Z}}$$

forms a strictly generating family of objects of $\mathcal{F}\mathcal{A}b$.

Corollary 3.1.9. *The category $\mathcal{F}\mathcal{A}b$ is an elementary closed quasi-abelian category. In particular, $\mathcal{F}\mathcal{A}b$ has enough projective objects. Moreover, for any projective object P of $\mathcal{F}\mathcal{A}b$, the functor*

$$P \otimes \cdot : \mathcal{F}\mathcal{A}b \to \mathcal{F}\mathcal{A}b$$

is strongly exact and $P \otimes P'$ is projective if P' is projective.

To show that $\mathcal{F}\mathcal{A}b$ has enough injective objects, we need first a few auxiliary results.

Definition 3.1.10. For any filtered abelian group M, we denote by $M(\infty)$ the filtered abelian group obtained by endowing M_∞ with the constant filtration.

Proposition 3.1.11. *Assume R is a cogenerator of $\mathcal{A}b$. Let FR denote the object of $\mathcal{F}\mathcal{A}b$ obtained by endowing the abelian group R with the filtration defined by setting*

$$FR_k = \begin{cases} R & \text{if } k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\prod_{k\in\mathbb{Z}\cup\{\infty\}} FR(k)$$

is a strict cogenerator of $\mathcal{FAb}$.

Proof. Let M be an arbitrary filtered abelian group. We have to show that the canonical morphism

$$i : M \longrightarrow \prod_{h\in\operatorname{Hom}_{\mathcal{FAb}}(M,\prod_{k\in\mathbb{Z}\cup\{\infty\}} FR(k))} (\prod_{k\in\mathbb{Z}\cup\{\infty\}} FR(k))$$

is a strict monomorphism. First, note that

$$\operatorname{Hom}_{\mathcal{FAb}}(M, \prod_{k\in\mathbb{Z}\cup\{\infty\}} FR(k)) = \prod_{k\in\mathbb{Z}\cup\{\infty\}} \operatorname{Hom}_{\mathcal{FAb}}(M, FR(k))$$

and that

$$\operatorname{Hom}_{\mathcal{FAb}}(M, FR(k))$$

is the subset of $\operatorname{Hom}(M_\infty, R)$ formed by morphisms $h_k : M \longrightarrow R$ such that

$$h_k(M_{-k-1}) = 0,$$

where we set $M_{-\infty} = 0$ by convention. Therefore, to give

$$h \in \operatorname{Hom}_{\mathcal{FAb}}(M, \prod_{k\in\mathbb{Z}} FR(k))$$

is to give a family

$$(h_k : M_\infty \longrightarrow R)_{k\in\mathbb{Z}}$$

of morphisms of abelian groups such that $h_k(M_{-k-1}) = 0$. Moreover, for any $m \in M$, we have

$$[i(m)_h]_k = h_k(m)$$

for any $h \in \operatorname{Hom}_{\mathcal{FAb}}(M, \prod_{k\in\mathbb{Z}} FR(k))$ and any $k \in \mathbb{Z}$.

Let us show that i is a monomorphism. Assume $m \in M_\infty \setminus \{0\}$. Since R is a cogenerator of $\mathcal{Ab}$, we can find a morphism

$$h_\infty : M_\infty \longrightarrow R$$

such that $h_\infty(m) \neq 0$. Setting $h_k = 0$ for any $k \in \mathbb{Z}$, we get a morphism

$$h \in \operatorname{Hom}_{\mathcal{FAb}}(M, \prod_{k\in\mathbb{Z}} FR(k))$$

such that
$$[i(m)_h]_\infty \neq 0.$$
Hence, $i(m) \neq 0$ and the conclusion follows.

Let us now prove that i is strict. Assume $m \in M_\infty$ is such that $i(m)$ has degree less than l. This means that
$$[i(m)_h]_k \in FR_{k+l}$$
for any $h \in \prod \operatorname{Hom}_{\mathcal{FAb}}(M, \prod_{k \in \mathbb{Z}\cup\{\infty\}} FR(k))$ and any $k \in \mathbb{Z}$. Therefore,
$$h_k(m) = 0$$
for any $k < -l$. Assume $m \notin M_l$. Denote $p : M_\infty \longrightarrow M_\infty/M_l$ the canonical projection. Since
$$p(m) \neq 0,$$
we can find a morphism $h' : M_\infty/M_l \longrightarrow R$ such that $h'(m) \neq 0$. Consider the morphism
$$h \in \operatorname{Hom}_{\mathcal{FAb}}(M, \prod_{k \in \mathbb{Z}\cup\{\infty\}} FR(k))$$
defined by setting
$$h_k = \begin{cases} h' \circ p & \text{if } k = -l-1 \\ 0 & \text{otherwise.} \end{cases}$$
We get a contradiction since
$$h_{-l-1}(m) \neq 0$$
and $-l-1 < -l$. Therefore, $m \in M_l$ and the proof is complete. $\square$

Proposition 3.1.12. *Assume I is an injective object of $\mathcal{Ab}$. Then,*
$$FI(k)$$
is an injective object of $\mathcal{FAb}$ for any $k \in \mathbb{Z}\cup\{\infty\}$.

Proof. Let M be an object of $\mathcal{FAb}$. For $k = \infty$, we have
$$\operatorname{Hom}_{\mathcal{FAb}}(M, FI(\infty)) = \operatorname{Hom}_{\mathcal{Ab}}(M_\infty, I)$$
and the result is obvious. Let us assume $k \neq \infty$. In this case, we have
$$\operatorname{Hom}_{\mathcal{FAb}}(M, FI(k)) = \operatorname{Hom}_{\mathcal{Ab}}(M_\infty/M_{-k-1}, I).$$
Let
$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a strictly exact sequence in $\mathcal{FA}b$. We get a commutative diagram of $\mathcal{A}b$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M'_{-k-1} \arrow[r] \arrow[d] & M_{-k-1} \arrow[r] \arrow[d] & M''_{-k-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M'_\infty \arrow[r] \arrow[d] & M_\infty \arrow[r] \arrow[d] & M''_\infty \arrow[r] \arrow[d] & 0 \\
 & M'_\infty/M'_{-k-1} \arrow[r] \arrow[d] & M_\infty/M_{-k-1} \arrow[r] \arrow[d] & M''_\infty/M''_{-k-1} \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where all the columns and the first two lines are exact. Therefore, the last line is also exact. Since I is injective in $\mathcal{A}b$, the sequence

$$0 \longleftarrow \operatorname{Hom}(M'_\infty/M'_{-k-1}, I) \longleftarrow \operatorname{Hom}(M_\infty/M_{-k-1}, I) \longleftarrow \operatorname{Hom}(M''_\infty/M''_{-k-1}, I) \longleftarrow 0$$

is exact. This shows that $FI(k)$ is injective in $\mathcal{FA}b$. □

Corollary 3.1.13. *The category $\mathcal{FA}b$ has enough injective objects.*

Proof. Apply the preceding propositions to an injective cogenerator of $\mathcal{A}b$ (e.g. $\mathbb{Q}/\mathbb{Z}$). □

3.1.2 Separated filtered abelian groups

Let us recall that any filtered abelian group M may be turned canonically into a topological abelian group by taking $\{M_k : k \in \mathbb{Z}\}$ as a fundamental system of neighborhoods of 0 in M_∞. In the sequel, when we apply topological vocabulary to a filtered abelian group, we always have this particular topological structure in mind. In particular, a filtered abelian group M is separated if and only if $\bigcap_{k\in\mathbb{Z}} M_k = 0$.

Definition 3.1.14. We denote $\widehat{\mathcal{FA}b}$ the full subcategory of $\mathcal{FA}b$ formed by separated filtered abelian groups.

Proposition 3.1.15. *Let $(M_i)_{i\in I}$ be a family of separated filtered abelian groups. Then, the filtered abelian groups*

$$\bigoplus_{i\in I} M_i \qquad \text{and} \qquad \prod_{i\in I} M_i$$

are separated. In particular, they form the direct sum and direct product of the family $(M_i)_{i\in I}$ in $\widehat{\mathcal{FA}b}$.

Proposition 3.1.16. *The category $\widehat{\mathcal{F}Ab}$ has kernels and cokernels. More precisely, let $u : M \longrightarrow N$ be a morphism of $\widehat{\mathcal{F}Ab}$. Then,*

(a) $\mathrm{Ker} u$ *is the subgroup* $u_\infty^{-1}(0)$ *endowed with the filtration*

$$(u_\infty^{-1}(0) \cap M_k)_{k\in\mathbb{Z}},$$

(b) $\mathrm{Coker} u$ *is the group* $N_\infty/\overline{u_\infty(M)}$ *endowed with the filtration*

$$(N_k + \overline{u_\infty(M)}/\overline{u_\infty(M)})_{k\in\mathbb{Z}}.$$

Hence,

(c) $\mathrm{Im} u$ *is the group* $\overline{u_\infty(M)}$ *endowed with the filtration*

$$(\overline{u_\infty(M)} \cap N_k)_{k\in\mathbb{Z}},$$

(d) $\mathrm{Coim} u$ *is the group* $M_\infty/u_\infty^{-1}(0)$ *endowed with the filtration*

$$(M_k + u_\infty^{-1}(0)/u_\infty^{-1}(0))_{k\in\mathbb{Z}}.$$

It is isomorphic to the group $u_\infty(M_\infty)$ *endowed with the filtration*

$$(u_\infty(M_k))_{k\in\mathbb{Z}}.$$

In particular, u is strict in $\widehat{\mathcal{F}Ab}$ if and only if it is strict in $\mathcal{F}Ab$ and has a closed range.

Proposition 3.1.17. *The category $\widehat{\mathcal{F}Ab}$ is quasi-abelian.*

Proof. We know that $\mathcal{F}Ab$ is quasi-abelian. By the characterization of the strict epimorphisms in $\widehat{\mathcal{F}Ab}$ and the structure of kernels in $\mathcal{F}Ab$, the axiom (QA) is automatically satisfied. Let us deal with the axiom (QA*). Consider the co-cartesian square

$$\begin{tikzcd} M' \arrow[r, "u'"] & N' \\ M \arrow[u, "v"] \arrow[r, "u"'] & N \arrow[u, "v'"'] \end{tikzcd}$$

of $\widehat{\mathcal{F}Ab}$. Assume u is a strict monomorphism of $\widehat{\mathcal{F}Ab}$. This means that u is a strict monomorphism of $\mathcal{F}Ab$ and that its range is closed. We know that

$$\begin{aligned} w : M &\longrightarrow N \oplus M' \\ m &\mapsto (u(m), v(m)) \end{aligned}$$

is a strict monomorphism of $\mathcal{FA}b$. Let $(m_k)_{k\in\mathbb{N}}$ be a sequence of M such that

$$w(n_k) \longrightarrow (n, m')$$

in $N \oplus M'$, it follows that

$$u(m_k) \longrightarrow n$$

in N. Since u is a strict monomorphism with closed range, there is $m \in M$ such that

$$m_k \longrightarrow m$$

in M and $u(m) = n$. Therefore,

$$v(m_k) \longrightarrow v(m)$$

and since M' is separated, we get

$$v(m) = m'.$$

Hence,

$$w(m) = (n, m')$$

and we see that the range of w is closed. Therefore, the sequence

$$0 \longrightarrow M \longrightarrow N \oplus M' \longrightarrow N' \longrightarrow 0$$

is strictly exact in $\mathcal{FA}b$. It follows that u' is a strict monomorphism of $\mathcal{FA}b$ and that

$$\mathrm{Coker} u \simeq \mathrm{Coker} u'.$$

Since $\mathrm{Coker} u$ is separated, $\mathrm{Coker} u'$ is also separated and u' has a closed range. $\square$

Proposition 3.1.18. *In* $\widehat{\mathcal{FA}b}$*,*

$$(\mathrm{F}\mathbb{Z}(l))_{l\in\mathbb{Z}}$$

forms a small strictly generating family of small projective objects. In particular, $\widehat{\mathcal{FA}b}$ *is a quasi-elementary quasi-abelian category.*

Proof. This follows directly from Proposition 3.1.15 and Proposition 3.1.16 since $\mathrm{F}\mathbb{Z}$ is clearly separated. $\square$

Remark 3.1.19. The object $\mathrm{F}\mathbb{Z}$ is not tiny since there are filtering inductive system of $\widehat{\mathcal{FA}b}$ with a non zero inductive limit in $\mathcal{FA}b$ but a zero inductive limit in $\widehat{\mathcal{FA}b}$. As an example, consider an object M of $\widehat{\mathcal{FA}b}$ and the system

$$(M(l))_{l\in\mathbb{Z}}.$$

In $\mathcal{FA}b$, we see easily that

$$\varinjlim_{l\in\mathbb{Z}} M(l)$$

is the group M_∞ with the constant filtration. Hence, in $\widehat{\mathcal{FA}b}$,

$$\varinjlim_{l\in\mathbb{Z}} M(l) \simeq 0.$$

Definition 3.1.20. We denote by

$$\widehat{I} : \widehat{\mathcal{FA}b} \longrightarrow \mathcal{FA}b$$

the canonical inclusion functor. We denote by

$$\widehat{L} : \mathcal{FA}b \longrightarrow \widehat{\mathcal{FA}b}$$

the functor defined by

$$\widehat{L}(M) = M/\cap_{k\in\mathbb{Z}} M_k.$$

Proposition 3.1.21. *There is a canonical adjunction isomorphism*

$$\mathrm{Hom}_{\mathcal{FA}b}(M, \widehat{I}(N)) \simeq \mathrm{Hom}_{\widehat{\mathcal{FA}b}}(\widehat{L}(M), N).$$

In particular, the functor $\widehat{I}$ is compatible with projective limits and the functor $\widehat{L}$ is compatible with inductive limits.

Proposition 3.1.22. *The functor*

$$\widehat{I} : \widehat{\mathcal{FA}b} \longrightarrow \mathcal{FA}b$$

is strictly exact and induces an equivalence of categories

$$\widehat{I} : \mathcal{D}(\widehat{\mathcal{FA}b}) \longrightarrow \mathcal{D}(\mathcal{FA}b).$$

Its quasi-inverse is given by

$$L\widehat{L} : \mathcal{D}(\mathcal{FA}b) \longrightarrow \mathcal{D}(\widehat{\mathcal{FA}b}).$$

Through this equivalence, the left t-structure of $\mathcal{D}(\widehat{\mathcal{FA}b})$ is exchanged with the left t-structure of $\mathcal{D}(\mathcal{FA}b)$. In particular,

$$\widehat{I} : \mathcal{LH}(\widehat{\mathcal{FA}b}) \approx \mathcal{LH}(\mathcal{FA}b).$$

Proof. Since any projective object of $\mathcal{F}\mathcal{A}b$ is a direct factor of a filtered abelian group of the form

$$\bigoplus_{i\in I} \mathrm{F}\mathbb{Z}(l_i)$$

which is separated, it is also separated. Let $M^\cdot$ be an object of $D(\widehat{\mathcal{F}\mathcal{A}b})$ and let

$$P^\cdot \xrightarrow{\sim} \widehat{I}(M^\cdot)$$

be a projective resolution of $\widehat{I}(M^\cdot)$. Since the components of $P^\cdot$ are separated,

$$\widehat{L}(P^\cdot) \simeq P^\cdot \simeq M^\cdot$$

in $D(\widehat{\mathcal{F}\mathcal{A}b})$. Hence,

$$L\widehat{L} \circ \widehat{I} \simeq \mathrm{id}\,.$$

Let

$$P^\cdot \simeq M^\cdot$$

be a projective resolution of $M^\cdot \in D(\mathcal{F}\mathcal{A}b)$. Since the components of $P^\cdot$ are separated, we have

$$\widehat{L}(P^\cdot) \simeq P^\cdot$$

in $D(\widehat{\mathcal{F}\mathcal{A}b})$ and

$$\widehat{I} \circ \widehat{L}(P^\cdot) \simeq M^\cdot$$

in $D(\mathcal{F}\mathcal{A}b)$. Hence,

$$\widehat{I} \circ L\widehat{L} \simeq \mathrm{id}\,.$$

To conclude, it is sufficient to remark that a sequence

$$M' \longrightarrow M \longrightarrow M''$$

of $\widehat{\mathcal{F}\mathcal{A}b}$ is strictly exact in $\mathcal{F}\mathcal{A}b$ if and only if it is strictly exact in $\widehat{\mathcal{F}\mathcal{A}b}$. □

Remark 3.1.23. The functor

$$\widehat{I} : \mathcal{D}(\widehat{\mathcal{F}\mathcal{A}b}) \longrightarrow \mathcal{D}(\mathcal{F}\mathcal{A}b)$$

does not preserve the right t-structures. As a matter of fact, if $u : M \longrightarrow N$ is a strict monomorphism of $\mathcal{F}\mathcal{A}b$ with a non closed range, the complex

$$0 \longrightarrow M \xrightarrow{u} N \longrightarrow 0$$

with M in degree 0 has null cohomology in that degree in $\mathcal{RH}(\mathcal{F}\mathcal{A}b)$ but not in $\mathcal{RH}(\widehat{\mathcal{F}\mathcal{A}b})$.

3.1.3 The category $\mathcal{R}$ and filtered sheaves

Definition 3.1.24. Let $\mathrm{R}\mathbb{Z}$ denote the graded ring $\mathbb{Z}[T]$.

To any filtered abelian group M, we associate the graded $\mathrm{R}\mathbb{Z}$-module

$$R(M) = \bigoplus_{k \in \mathbb{Z}} M_k,$$

the multiplication

$$\cdot T : M_k \longrightarrow M_{k+1}$$

being the canonical inclusion. This gives us an additive functor

$$R : \mathcal{FAb} \to \mathcal{M}od(\mathrm{R}\mathbb{Z})$$

where $\mathcal{M}od(\mathrm{R}\mathbb{Z})$ denotes the category of (graded) $\mathrm{R}\mathbb{Z}$-modules.

Let N be an $\mathrm{R}\mathbb{Z}$-module. Denote

$$n_{k+1,k} : N_k \longrightarrow N_{k+1}$$

the action of T and consider the inductive system

$$(N_k, n_{k+1,k})_{k \in \mathbb{Z}}.$$

Set

$$(L(N))_\infty = \varinjlim_{k \in \mathbb{Z}} N_k$$

and let $(L(N))_k$ be the canonical image of N_k in $(L(N))_\infty$. Clearly,

$$(L(N)_k)_{k \in \mathbb{Z}}$$

forms a filtration of the abelian group $(L(N))_\infty$. We denote $L(N)$ the corresponding filtered abelian group. This gives us an additive functor

$$L : \mathcal{M}od(\mathrm{R}\mathbb{Z}) \longrightarrow \mathcal{FAb}.$$

Proposition 3.1.25. *We have the canonical functorial isomorphisms*

$$\mathrm{Hom}_{\mathcal{FAb}}(L(N), M) \simeq \mathrm{Hom}_{\mathcal{M}od(\mathrm{R}\mathbb{Z})}(N, R(M))$$

and

$$L \circ R(M) \simeq M.$$

In particular, R is a fully faithful continuous functor and L is a cocontinuous functor. Moreover, R is strictly exact and is compatible with direct sums.

Proposition 3.1.26. *The essential image of*

$$R : \mathcal{FA}b \longrightarrow \mathcal{M}od(\mathrm{R}\mathbb{Z})$$

is formed by $\mathrm{R}\mathbb{Z}$*-modules* M *such that*

$$T\cdot : M \longrightarrow M$$

is injective.

Proposition 3.1.27. *The essential image of R forms a L-projective subcategory of $\mathcal{M}od(\mathrm{R}\mathbb{Z})$. In particular,*

$$L : \mathcal{M}od(\mathrm{R}\mathbb{Z}) \longrightarrow \mathcal{FA}b$$

is an explicitly left derivable right exact functor which has finite homological dimension.

Proof. Let us denote $\mathcal{P}$ the essential image of R.

(a) Any object of $\mathcal{M}od(\mathrm{R}\mathbb{Z})$ is a quotient of an object of $\mathcal{P}$. As a matter of fact, the canonical morphism

$$\bigoplus_{k\in\mathbb{Z}} \bigoplus_{h\in M_k} \mathrm{R}\mathbb{Z}(-k) \longrightarrow M$$

is an epimorphism for any $\mathrm{R}\mathbb{Z}$-module M and it follows from the preceding propositions that

$$\bigoplus_{k\in\mathbb{Z}} \bigoplus_{h\in M_k} \mathrm{R}\mathbb{Z}(-k) \simeq R(\bigoplus_{k\in\mathbb{Z}} \bigoplus_{h\in M_k} \mathrm{F}\mathbb{Z}(-k)).$$

(b) In an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of $\mathcal{M}od(\mathrm{R}\mathbb{Z})$ where M, M'' belong to $\mathcal{P}$, M belongs to $\mathcal{P}$. This follows directly from the preceding proposition since a subobject of an object of $\mathcal{P}$ is clearly an object of $\mathcal{P}$.

(c) If

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is an exact sequence of $\mathcal{M}od(\mathrm{R}\mathbb{Z})$ with M', M, M'' in $\mathcal{P}$, the sequence

$$0 \longrightarrow L(M') \longrightarrow L(M) \longrightarrow L(M'') \longrightarrow 0$$

is strictly exact in $\mathcal{FA}b$. As a matter of fact, we may assume

$$M' \simeq R(N'), \quad M \simeq R(N), \quad M'' \simeq R(N'').$$

Hence, using the fact that R is fully faithful, we see that the given exact sequence may be obtained by applying R to a strictly exact sequence of the form

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

of $\mathcal{FA}b$. The conclusion follows from the fact that $L \circ R \simeq \mathrm{id}$. $\square$

Proposition 3.1.28. *The functor*

$$RR : \mathcal{D}(\mathcal{FA}b) \longrightarrow \mathcal{D}(\mathcal{M}od(\mathrm{R}\mathbb{Z}))$$

is an equivalence of categories which exchanges the left t-structure of $\mathcal{D}(\mathcal{FA}b)$ with the canonical t-structure of $\mathcal{D}(\mathcal{M}od(\mathrm{R}\mathbb{Z}))$. A quasi-inverse of RR is given by

$$LL : \mathcal{D}(\mathcal{M}od(\mathrm{R}\mathbb{Z})) \longrightarrow \mathcal{D}(\mathcal{FA}b).$$

In particular, R induces an equivalence of categories

$$\mathcal{LH}(\mathcal{FA}b) \approx \mathcal{M}od(\mathrm{R}\mathbb{Z}).$$

Proof. This follows directly from the preceding propositions. $\square$

Corollary 3.1.29. *The functor*

$$\mathcal{FA}b \longrightarrow \mathcal{A}b^{\mathbb{Z}}$$

which associates to any filtered abelian group M the inductive system

$$k \mapsto M_k,$$

the transitions of which are given by the inclusions

$$M_k \longrightarrow M_{k'} \qquad (k \leq k'),$$

induces an equivalence of categories

$$\mathcal{LH}(\mathcal{FA}b) \approx \mathcal{A}b^{\mathbb{Z}}.$$

Proof. This follows from the preceding proposition if one notes that the functor

$$\mathcal{M}od(\mathrm{R}\mathbb{Z}) \longrightarrow \mathcal{A}b^{\mathbb{Z}}$$

which sends an $\mathrm{R}\mathbb{Z}$-module M to the inductive system

$$(M_k, T\cdot : M_k \longrightarrow M_{k'})_{k \in \mathbb{Z}}$$

is clearly an equivalence of categories. $\square$

Proposition 3.1.30. *The structure of closed category of $\mathcal{F}Ab$ induces a structure of closed category on $\mathcal{LH}(\mathcal{F}Ab)$ which is compatible with the usual structure of closed category of $\mathcal{M}od(\mathrm{R}\mathbb{Z})$ through the equivalence of the preceding proposition.*

Proof. Note that there are obvious canonical functorial morphisms

$$R(M) \otimes R(N) \to R(M \otimes N)$$

$$R(\mathrm{FHom}\,(M, N)) \to \mathrm{GHom}\,(R(M), R(N))$$

for M, N in $\mathcal{F}Ab$. Although they are not bijective in general, one can check easily that they become isomorphisms if $M \simeq \mathrm{F}\mathbb{Z}(l)$ for some $l \in \mathbb{Z}$. Therefore, we see that

$$RR(M) \otimes^L RR(N) \simeq RR(M \otimes^L N)$$

$$RR(\mathrm{RFHom}(M, N)) \simeq \mathrm{RGHom}(RR(M), RR(N))$$

and the conclusion follows. □

Remark 3.1.31. The category $\mathcal{R} = \mathcal{LH}(\mathcal{F}Ab)$ is a closed elementary abelian category for which we may apply all the results obtained in the preceding chapter. In particular, the cohomological properties of filtered sheaves on a topological space X are best understood by working in $\mathcal{D}(X, \mathcal{R})$.

3.2 Topological Sheaves

3.2.1 The category of semi-normed spaces

In this section all vector spaces are $\mathbb{C}$-vector spaces. Recall that a semi-norm on a vector space E is a positive function p on E such that

$$\begin{aligned} p(e_1 + e_2) \leq p(e_1) + p(e_2) &\quad \text{for any} \quad e_1, e_2 \in E \\ p(ce) \leq |c| p(e) &\quad \text{for any} \quad c \in \mathbb{C}, e \in E \end{aligned}$$

Let E be a vector subspace of F and let p be a semi-norm on F. Recall that p induces a semi-norm p' on E and a semi-norm p'' on F/E. These semi-norms are defined respectively by

$$p'(e) = p(e) \quad \text{and} \quad p''([f]_E) = \inf_{e \in E} p(f + e).$$

Definition 3.2.1. A *semi-normed space*, is a vector space endowed with a semi-norm p_E. A *morphism of semi-normed spaces* is a morphism $f : E \longrightarrow F$ of the underlying vector spaces such that

$$|p_F \circ f| \leq C p_E$$

for some $C > 0$. With this notion of morphisms, semi-normed spaces form a category which we denote by $\mathcal{S}ns$.

Let E be a semi-normed space. As is well-known, the semi-norm p_E gives rise to a canonical locally convex topology on E. Hereafter, we will always have this particular topology in mind when we use topological vocabulary in relation with semi-normed spaces. Using this convention, a morphism of semi-normed spaces is simply a continuous linear map.

Lemma 3.2.2. *The category $\mathcal{S}ns$ is additive. More precisely:*

(a) *The $\mathbb{C}$-vector space 0 endowed with the 0 semi-norm is a null object of $\mathcal{S}ns$.*

(b) *For any E,F in $\mathcal{S}ns$, the $\mathbb{C}$-vector space $E \oplus F$ endowed with the semi-norm p defined by*

$$p(e, f) = p_E(e) + p_F(f)$$

is a biproduct of E and F in $\mathcal{S}ns$.

Lemma 3.2.3. *Let $f : E \longrightarrow F$ be a morphism of $\mathcal{S}ns$. Then,*

(a) $\mathrm{Ker} f$ *is the vector space $f^{-1}(0)$ endowed with the semi-norm induced by p_E;*

(b) $\mathrm{Coker} f$ *is the vector space $F/f(E)$ endowed with semi-norm induced by p_F.*

Therefore,

(c) $\mathrm{Im} f$ *is the vector space $f(E)$ endowed with the semi-norm induced by p_F;*

(d) $\mathrm{Coim} f$ *is the vector space $E/f^{-1}(0)$ endowed with the semi-norm induced by p_E. Equivalently,* $\mathrm{Coim} f$ *may be described as the vector space $f(E)$ endowed with the semi-norm p defined by*

$$p(x) = \inf_{y \in f^{-1}(x)} p_E(y)$$

for any $x \in f(E)$.

In particular,

(e) f is strict if and only if

$$\inf_{y \in f^{-1}(0)} p_E(x+y) \leq C p_F(f(x))$$

for some $C > 0$. In other words, f is strict if and only if it is relatively open.

Proposition 3.2.4. *The category $\mathcal{S}ns$ is quasi-abelian.*

Proof. We know that $\mathcal{S}ns$ is additive and that any morphism of $\mathcal{S}ns$ has a kernel and a cokernel.

Consider the cartesian square

$$\begin{tikzcd} E \arrow[r, "f"] & F \\ T \arrow[u, "v"] \arrow[r, "u"'] & G \arrow[u, "g"'] \end{tikzcd}$$

where f is a strict epimorphism and let us show that u is a strict epimorphism. We may assume that T is the kernel of

$$E \oplus G \xrightarrow{(\,f \ -g\,)} F.$$

Hence,

$$T = \{(x, y) \in E \oplus G : f(x) = g(y)\}.$$

Denoting

$$i : \mathrm{Ker}\left(f \quad -g\right) \to E \oplus G$$

the canonical injection and π_E and π_G the canonical projections, we have

$$v = \pi_E \circ i \quad \text{and} \quad u = \pi_G \circ i.$$

Since f is surjective, for any $y \in G$ there is $x \in E$ such that

$$f(x) = g(y).$$

In such a case, $(x, y) \in T$ and

$$u(x, y) = y.$$

This shows that the application u is surjective.

Recall that

$$p_{E \oplus G}((x, y)) = p_E(x) + p_G(y) \qquad \forall (x, y) \in E \oplus G.$$

Clearly,

$$\mathrm{Ker}u = \{(e,0) \in E \oplus G : f(e) = 0\}.$$

Therefore, for any $(x, y) \in T$, we have

$$\inf_{(e,e')\in \mathrm{Ker}u} p((x,y) + (e,e')) = \inf_{e\in \mathrm{Ker}f} p_E(x+e) + p_G(y).$$

Since f is a strict epimorphism, there is $C > 0$ such that

$$\inf_{e\in \mathrm{Ker}f} p_E(x+e) \leq C p_F(f(x))$$

for any $x \in E$. From the continuity of g, we get $C' > 0$ such that

$$p_F(f(x)) = p_F(g(y)) \leq C' p_G(y).$$

for any $(x, y) \in T$. Hence, there is $C'' > 0$ such that

$$\inf_{(e,e')\in \mathrm{Ker}u} p((x,y) + (e,e')) \leq C'' p_G(y)$$

for any $(x, y) \in T$ and u is a strict epimorphism.

Consider the cocartesian square

$$\begin{tikzcd} G \arrow[r, "u"] & T \\ E \arrow[u, "g"] \arrow[r, "f"'] & F \arrow[u, "v"'] \end{tikzcd}$$

where f is a strict monomorphism and let us show that u is a strict monomorphism. Denote α the morphism

$$\begin{pmatrix} g \\ -f \end{pmatrix} : E \longrightarrow G \oplus F.$$

We may assume that

$$T = \mathrm{Coker}\alpha = (G \oplus F)/\alpha(E).$$

Denoting

$$q : G \oplus F \longrightarrow (G \oplus F)/\alpha(E)$$

the canonical morphism and σ_F and σ_G the canonical embeddings, we have

$$u = q \circ \sigma_G \quad \text{and} \quad v = q \circ \sigma_F.$$

Consider $y \in G$ such that $u(y) = q \circ \sigma_G(y) = 0$. It follows that

$$(y, 0) \in \alpha(E)$$

and there is $x \in E$ such that

$$(y, 0) = (g(x), -f(x)).$$

Since f is injective, $x = 0$ and we get $y = g(x) = 0$. Hence u is injective.

Since g is continuous and f is strict, we can find positive constants C, C' and C'' such that

$$\begin{aligned} p_G(y) &\leq p_G(y + g(x)) + p_G(g(x)) \\ &\leq p_G(y + g(x)) + C p_E(x) \\ &\leq p_G(y + g(x)) + C' p_F(f(x)) \\ &\leq C'' p_{G\oplus F}((y + g(x), -f(x))) \end{aligned}$$

for all $y \in G$ and all $x \in E$. Therefore, for any $y \in G$, we have

$$\begin{aligned} p_G(y) &\leq C'' \inf_{x \in E} p((y + g(x), -f(x))) \\ &\leq C'' \inf_{(y',z') \in \alpha(E)} p((y + y', z')) \\ &\leq C'' \tilde{p}(q((y, 0))) \\ &\leq C'' \tilde{p}(u(y)), \end{aligned}$$

where $\tilde{p}$ denotes the semi-norm of $(G \oplus F)/\alpha(E)$ induced by p. It follows that u is a strict monomorphism. $\square$

Definition 3.2.5. Let E and F be semi-normed spaces.

We denote by $E \otimes F$ the semi-normed space obtained by endowing the vector space $E \otimes_{\mathbb{C}} F$ with the semi-norm p defined by

$$p(z) = \inf_{z = \sum_{k=1}^{N} x_k \otimes y_k} p_E(x_k) p_F(y_k).$$

We denote by $\mathrm{L}\,(E, F)$ the vector space $\mathrm{Hom}_{\mathbb{C}}(E, F)$ endowed with the semi-norm q defined by

$$q(h) = \sup_{p_E(x) \leq 1} p_F(h(x)).$$

We denote by $\mathbb{C}$ the semi-normed space obtained by endowing $\mathbb{C}$ with the semi-norm $|\cdot|$.

Proposition 3.2.6. *The category $\mathcal{S}ns$ endowed with $\otimes$ as internal tensor product, L as internal homomorphisms functor and $\mathbb{C}$ as unit object form a closed category.*

Definition 3.2.7. Let $(E_i)_{i\in I}$ be a family of semi-normed spaces. We denote by $\overset{\smile}{\bigoplus\limits_{i\in I}} E_i$ the vector space $\bigoplus_{i\in I} E_i$ endowed with the semi-norm p defined by

$$p((e_i)_{i\in I}) = \sum_{i\in I} p_i(e_i).$$

We denote by $\overset{\smile}{\prod\limits_{i\in I}} E_i$ the subvector space of $\prod_{i\in I} E_i$ formed by the families $(e_i)_{i\in I}$ such that

$$p(e) = \sup_{i\in I} p_i(e_i) < +\infty$$

endowed with the semi-norm p.

Proposition 3.2.8. *(a) Let $(u_i : E_i \longrightarrow F)_{i\in I}$ be a bounded family of morphisms of semi-normed spaces. Then, there is a unique morphism*

$$u : \overset{\smile}{\bigoplus_{i\in I}} E_i \longrightarrow F$$

such that $u \circ s_i = u_i$ (here $s_i : E_i \longrightarrow \overset{\smile}{\bigoplus\limits_{i\in I}} E_i$ denotes the canonical monomorphism).

(b) Let $(v_i : F \longrightarrow E_i)_{i\in I}$ be a bounded family of morphisms of semi-normed spaces. Then, there is a unique morphism

$$v : F \longrightarrow \overset{\smile}{\prod_{i\in I}} E_i$$

such that $p_i \circ v = v_i$ (here $p_i : \overset{\smile}{\prod\limits_{i\in I}} \longrightarrow E_i$ denotes the canonical epimorphism).

(c) Let $(u_i : E_i \longrightarrow F_i)_{i\in I}$ be a bounded family of morphisms of semi-normed spaces. Then, the kernel and the cokernel of

$$\overset{\smile}{\bigoplus_{i\in I}} E_i \to \overset{\smile}{\bigoplus_{i\in I}} F_i$$

are respectively isomorphic to $\overset{\smile}{\bigoplus\limits_{i\in I}} \mathrm{Ker} u_i$ and $\overset{\smile}{\bigoplus\limits_{i\in I}} \mathrm{Coker} u_i$. Similarly, the kernel of

$$\overset{\smile}{\prod_{i\in I}} E_i \to \overset{\smile}{\prod_{i\in I}} F_i$$

is isomorphic to $\breve{\prod\limits_{i\in I}}\mathrm{Ker} u_i$. *Moreover, if each* u_i *is strict, then the cokernel of*

$$\breve{\prod_{i\in I}} E_i \to \breve{\prod_{i\in I}} F_i$$

is isomorphic to $\breve{\prod\limits_{i\in I}}\mathrm{Coker} u_i$.

Remark 3.2.9. We could also introduce the subcategory $\breve{\mathcal{S}}ns$ of $\mathcal{S}ns$ whose morphisms are the linear maps $f : E \longrightarrow F$ such that

$$|p_F \circ f| \le p_E.$$

Then,

$$\breve{\bigoplus_{i\in I}} \qquad \text{and} \qquad \breve{\prod_{i\in I}}$$

appear as the true direct sum and direct product of objects of $\breve{\mathcal{S}}ns$. Note however that $\breve{\mathcal{S}}ns$ is *not* additive and so does not enter into the general framework of quasi-abelian categories.

Corollary 3.2.10. *For any set* I

$$\breve{\bigoplus_{i\in I}} \mathbb{C} \qquad \textit{(resp. } \breve{\prod_{i\in I}} \mathbb{C} \textit{)}$$

is projective (resp. injective) in $\mathcal{S}ns$.

Proof. The first part follows directly from the preceding proposition thanks to the characterization of strict epimorphisms contained in Lemma 3.2.3. As for the second part, the characterization of strict monomorphisms (loc. cit.) reduces it to the well-known Hahn-Banach theorem. □

Proposition 3.2.11. *The category* $\mathcal{S}ns$ *has enough projective and injective objects.*

Proof. (a) For any object E of $\mathcal{S}ns$, the canonical morphism

$$\breve{\bigoplus_{b\in B_E}} \mathbb{C} \xrightarrow{u} E$$

defined by

$$u((c_b)_{b\in b_E}) = \sum_{b\in B_E} c_b b$$

is a strict epimorphism. As a matter of fact, any $b' \in B_E$ may be written as

$$u((\delta_{b'b})_{b\in B_E})$$

and

$$p_{\breve{\bigoplus}_{b\in B_E}\mathbb{C}}(\delta_{b'b})_{b\in B_E} = 1.$$

Thanks to the preceding corollary, it follows that $\mathcal{S}ns$ has enough projective object.

(b) Let E be an object of $\mathcal{S}ns$ and let us show that there is a strict monomorphism from E to an injective object of $\mathcal{S}ns$. Denote N the subspace $p^{-1}(0)$ of E endowed with the null semi-norm. Since any linear map $h : X \longrightarrow N$ is continuous, it is clear that N is injective in $\mathcal{S}ns$. Therefore, the sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow E/N \longrightarrow 0$$

splits in $\mathcal{S}ns$ and E is isomorphic to $N \oplus E/N$. Hence, we may assume $N = 0$ (i.e. E is separated). In this case, denote by B'_E the unit semi-ball of $\mathrm{L}\,(E, \mathbb{C})$ and consider the morphism

$$v : E \longrightarrow \breve{\prod_{\varphi\in B'_E}} \mathbb{C}$$

defined by

$$v(e)_\varphi = \varphi(e).$$

Thanks to the theorem of bipolars, this is clearly a strict monomorphism and the conclusion follows from the preceding corollary. $\square$

Proposition 3.2.12. *Let $(E_i)_{i\in I}$ be a family of semi-normed spaces. Then, for any semi-normed space F we have the following canonical isomorphisms*

$$\breve{\bigoplus_{i\in I}}(E_i \otimes F) \xrightarrow{\sim} (\breve{\bigoplus_{i\in I}} E_i) \otimes F$$

$$\mathrm{L}\,(\breve{\bigoplus_{i\in I}} E_i, F) \xrightarrow{\sim} \breve{\prod} \mathrm{L}\,(E_i, F)$$

$$\mathrm{L}\,(F, \breve{\prod_{i\in I}} E_i) \xrightarrow{\sim} \breve{\prod} \mathrm{L}\,(F, E_i)$$

Proof. This follows directly from the adjunction formula

$$\mathrm{Hom}_{\breve{\mathcal{S}ns}}(E \otimes F, G) \simeq \mathrm{Hom}_{\breve{\mathcal{S}ns}}(E, L(F, G)).$$

□

Proposition 3.2.13. *For any projective object P of $\mathcal{S}ns$ the functor*

$$P \otimes \cdot : \mathcal{S}ns \longrightarrow \mathcal{S}ns$$

is strongly exact. Moreover for any projective object P' of $\mathcal{S}ns$, the object $P \otimes P'$ is also projective.

Proof. Let P be a projective object of $\mathcal{S}ns$. Since the result will be true for a direct factor of P if it is true for P, we may assume that P is of the form $\breve{\bigoplus_{i\in I}} \mathbb{C}$. Thanks to Propositions 3.2.12 and 3.2.8 we may even reduce ourselves to the case $P = \mathbb{C}$. But $\mathbb{C}$ is the unit object of the closed category $\mathcal{S}ns$, so we get the conclusion. □

Corollary 3.2.14. *The abelian category $\mathcal{LH}(\mathcal{S}ns)$ has a canonical structure of closed category.*

Proof. This follows from Corollary 1.5.4. □

3.2.2 The category of normed spaces

Definition 3.2.15. We denote by $\mathcal{N}vs$ the full subcategory of $\mathcal{S}ns$ formed by normed vector spaces.

Proposition 3.2.16. *Let $u : E \longrightarrow F$ be a morphism of $\mathcal{N}vs$. Then*

(a) $\mathrm{Ker}\, u$ *is the subspace $u^{-1}(0)$ endowed with the norm induced by that of E,*

(b) $\mathrm{Coker}\, u$ *is the quotient space $F/\overline{u(E)}$ endowed with the norm induced by that of F,*

(c) $\mathrm{Im}\, u$ *is the subspace $\overline{u(E)}$ endowed with the norm induced by that of F,*

(d) $\mathrm{Coim}\, u$ *is the quotient space $E/u^{-1}(0)$ endowed with the norm induced by that of E,*

(e) u is strict if and only if u is relatively open with closed range.

Proof. Direct. □

Proposition 3.2.17. *The category $\mathcal{N}vs$ is quasi-abelian.*

Proof. Since $\mathcal{N}vs$ is clearly additive with kernels and cokernels, we only need to prove that axiom QA is satisfied. Let

$$\begin{tikzcd} E_0 \arrow[r, "u_0"] & F_0 \\ E_1 \arrow[u] \arrow[r, "u_1"'] & F_1 \arrow[u] \end{tikzcd}$$

be a cartesian square with u a strict epimorphism. It follows from the preceding proposition that u_0 is a strict epimorphism of $\mathcal{S}ns$ and that the square is cartesian in $\mathcal{S}ns$. Therefore, u_1 is a strict epimorphism in $\mathcal{S}ns$ and thus in $\mathcal{N}vs$. Now, let

$$\begin{tikzcd} E_1 \arrow[r, "u_1"] & F_1 \\ E_0 \arrow[u, "e"] \arrow[r, "u_0"'] & F_0 \arrow[u, "f"'] \end{tikzcd}$$

be a cocartesian square in $\mathcal{N}vs$ where u_0 is a strict monomorphism of $\mathcal{N}vs$. It follows that u_0 is a strict monomorphism of $\mathcal{S}ns$ with closed range. By definition the sequence

$$E_0 \xrightarrow{\varphi=\begin{pmatrix}-e\\ u_0\end{pmatrix}} E_1 \oplus F_0 \xrightarrow{\psi=\begin{pmatrix}u_1 & f\end{pmatrix}} F_1 \longrightarrow 0 \qquad (*)$$

is costrictly exact in $\mathcal{N}vs$. We know φ is a strict monomorphism of $\mathcal{S}ns$. Let us prove that its range is closed. Assume x_m is a sequence of E_0 such that

$$(-e(x_m), u_0(x_m)) \longrightarrow (y, z)$$

in $E_1 \oplus F_0$. Then, $u_0(x_m) \longrightarrow z$ in F_0 and since u_0 has closed range, there is x in E_0 such that $u_0(x_m) \longrightarrow u_0(x)$. Since u_0 is relatively open, $x_m \longrightarrow x$ in E_0. Therefore $(-e(x_m), u_0(x_m)) \longrightarrow (-e(x), u_0(x))$ and since $E_1 \oplus F_0$ is separated, we see that $(y, z) = (-e(x), u_0(x))$. Hence φ is a strict monomorphism of $\mathcal{N}vs$ and the sequence (*) is costrictly exact in $\mathcal{S}ns$. It follows that u_1 is a strict monomorphism of $\mathcal{S}ns$ an it remains to show that it has a closed range. Let x_m be a sequence of E_1 such that $u_1(x_m) \longrightarrow y$ in F_1. Set $y = \psi(z, t)$. This means that

$$\psi(x_m - z, -t) \longrightarrow 0$$

in F_1. Since the sequence (*) is strictly exact, there is a sequence s_m of E_0 such that

$$(x_m - z + e(s_m), -t - u_0(s_m)) \longrightarrow 0$$

in $E_1 \oplus F_0$. It follows that $u_0(s_m) \longrightarrow -t$ in F_0. Hence $s_m \longrightarrow s$ in E_0 and $u_0(s) = -t$. Moreover, $x_m \longrightarrow x = z - e(s)$ in E_1. Clearly, $u_1(x) = \psi(x, 0) = \psi((z, t) + (-e(s), u_0(s))) = y$ and the conclusion follows. $\square$

Proposition 3.2.18. *The canonical inclusion*

$$I : \mathcal{N}vs \longrightarrow \mathcal{S}ns$$

has a right adjoint

$$\mathrm{Sep} : \mathcal{S}ns \longrightarrow \mathcal{N}vs.$$

Moreover, $\mathrm{Sep} \circ I = \mathrm{id}_{\mathcal{N}vs}$.

Proof. We define Sep by setting

$$\mathrm{Sep}(E) = E/N$$

where $N = \{x \in E : p_E(x) = 0\}$. One checks easily that

$$\mathrm{Hom}_{\mathcal{S}ns}(E, I(F)) = \mathrm{Hom}_{\mathcal{N}vs}(E/N, F)$$

and the conclusion follows. □

Proposition 3.2.19. *The functor*

$$\mathrm{Sep} : \mathcal{S}ns \longrightarrow \mathcal{N}vs$$

is strongly right exact and has a left derived functor

$$\mathrm{LSep} : \mathcal{D}^*(\mathcal{S}ns) \longrightarrow \mathcal{D}^*(\mathcal{N}vs) \qquad * \in \{\emptyset, +, -, b).$$

The functor

$$I : \mathcal{N}vs \longrightarrow \mathcal{S}ns$$

is strictly exact and gives rise to a functor

$$I : \mathcal{D}^*(\mathcal{S}ns) \longrightarrow \mathcal{D}^*(\mathcal{N}vs) \qquad * \in \{\emptyset, +, -, b\}.$$

Moreover, I *and* LSep *define quasi-inverse equivalences of categories. In particular,*

$$I : \mathcal{LH}(\mathcal{N}vs) \longrightarrow \mathcal{LH}(\mathcal{S}ns)$$

is an equivalence of categories. (Note that the same result does not hold for $\mathcal{RH}(\mathcal{N}vs)$ *and* $\mathcal{RH}(\mathcal{S}ns)$*.)*

Proof. Since any object of $\mathcal{S}ns$ is a quotient of an object of the form

$$\overset{\smile}{\bigoplus_{i\in I}} \mathbb{C}$$

and since objects of this form are clearly separated, one sees easily that $\mathcal{N}vs$ forms a Sep-projective subcategory of $\mathcal{S}ns$. The conclusion follows. □

Corollary 3.2.20. *The functor*

$$\mathrm{Sep} : \mathcal{S}ns \to \mathcal{S}ns$$

is left derivable and

$$\mathrm{LSep} : \mathcal{D}^*(\mathcal{S}ns) \to \mathcal{D}^*(\mathcal{S}ns) \qquad * \in \{\emptyset, +, -, b\}$$

is isomorphic to the identity.

3.2.3 The category $\mathcal{W}$ and topological sheaves

In this section, we fix two universes U and V such that $U \in V$.

Proposition 3.2.21. *The category*

$$\mathcal{W} = \mathcal{I}nd_V\, \mathcal{S}ns_U$$

is an elementary quasi-abelian category. It has a canonical closed structure extending that of $\mathcal{S}ns_U$. For any projective object P of $\mathcal{I}nd_V\, \mathcal{S}ns_U$, the functor

$$P \otimes \cdot : \mathcal{I}nd_V(\mathcal{S}ns_U) \to \mathcal{I}nd_V(\mathcal{S}ns_U)$$

is strongly exact and transforms a projective object into a projective object. In particular,

$$\mathcal{LH}(\mathcal{I}nd_V(\mathcal{S}ns_U)) \approx \mathcal{I}nd_V(\mathcal{LH}(\mathcal{S}ns_U))$$

is canonically a closed abelian category, the projective objects of which have similar properties.

Proof. This is a direct consequence of Proposition 2.1.17, Proposition 2.1.19 and Proposition 1.5.4. □

Corollary 3.2.22. *Let $\mathcal{P}$ denote the full additive subcategory of $\mathcal{S}ns_U$ formed by semi-normed spaces of the form*

$$\overset{\smile}{\bigoplus_{i \in I}} \mathbb{C}$$

for some U-set I. Then, we have the canonical equivalence of categories

$$\mathcal{W} \approx \mathcal{A}dd(\mathcal{P}, \mathcal{A}b_V).$$

Proof. This follows from Proposition 2.1.14. □

Definition 3.2.23. We denote $\mathcal{T}c$ the category formed by locally convex topological vector spaces and continuous linear maps. For any object E of $\mathcal{T}c$, we denote by $\mathcal{B}_E$ the ordered set formed by absolutely convex bounded susbsets. For any $B \in \mathcal{B}_E$, we denote E_B the vector subspace of E generated by B endowed with the gauge semi-norm associated to B.

Proposition 3.2.24. *The functor*

$$W : \mathcal{T}c_V \to \mathcal{W}$$

defined by setting

$$W(E) = \varinjlim_{B \in \mathcal{B}_E} \text{“}E_B\text{”}$$

is faithfull. Moreover, $\mathrm{Hom}_{\mathcal{W}}(W(E), W(F))$ *is formed by linear maps from E to F which transform bounded subsets of E into bounded subsets of F. In particular,*

$$\mathrm{Hom}_{\mathcal{W}}(W(E), W(F)) = \mathrm{Hom}_{\mathcal{T}c}(E, F)$$

if E is bornological.

Proof. This follows directly from the formula

$$\mathrm{Hom}_{\mathcal{W}}(\varinjlim_{B \in \mathcal{B}_E} \text{“}E_B\text{”}, \varinjlim_{B' \in \mathcal{B'}_F} \text{“}F_{B'}\text{”}) \simeq \varprojlim_{B \in \mathcal{B}_E} \varinjlim_{B' \in \mathcal{B'}_F} \mathrm{Hom}_{\mathcal{S}ns}(E_B, F_{B'}).$$

□

Remark 3.2.25. Let E be an object of $\mathcal{T}c_V$. Through the equivalence of Corollary 3.2.22, $W(E)$ corresponds to the functor

$$P \mapsto \mathrm{Hom}_{\mathcal{T}c}(P, E)$$

from $\mathcal{P}$ to $\mathcal{A}b_V$. Note also that

$$\mathrm{Hom}_{\mathcal{T}c}(\overset{\smile}{\bigoplus_{i \in I}} \mathbb{C}, E) \simeq \ell_\infty(I; E)$$

where $\ell_\infty(I; E)$ denotes the space of bounded families of E which are indexed by I.

Proposition 3.2.26. *The functor*

$$W : \mathcal{T}c_U \to \mathcal{W}$$

preserves projective limits. Moreover, an algebraically exact sequence

$$0 \to E' \to E \to E'' \to 0$$

of FN (resp. DFN) spaces gives rise to the exact sequence

$$0 \to W(E') \to W(E) \to W(E'') \to 0$$

of $\mathcal{W}$.

Proof. This follows directly from the preceding remark combined with well-known results of functional analysis. $\square$

Remark 3.2.27. The preceding result show that a sheaf with values in $\mathcal{T}c_U$ give rise to a sheaf with values in $\mathcal{W}$. Note also that the categories of FN and DFN spaces appear as full subcategories of $\mathcal{W}$. Since we may apply to the category $\mathcal{W}$ all the results of the preceding chapter, the cohomological theory of $\mathcal{W}$-sheaves is well-behaved. Putting all these fact together make us feel that $\mathcal{W}$-sheaves form a convenient class of topological sheaves for applications to algebraic analysis.

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