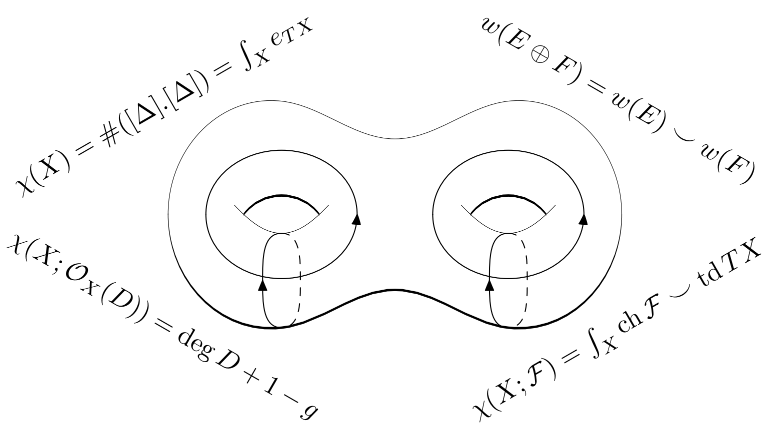
TEXTOS DE MATEMÁTICA

INTRODUCTION TO CHARACTERISTIC CLASSES AND INDEX THEORY

Jean-Pierre Schneiders



UNIVERSIDADE DE LISBOA
Faculdade de Ciências
Departamento de Matemática

Textos de Matemática, Volume 13,

Departamento de Matemática
Faculdade de Ciências da Universidade de Lisboa, 2000

Editores: Fernando C. Silva e L. Trabucho

Título: *Introduction to Characteristic Classes and Index Theory*

Autor: Jean-Pierre Schneiders

ISBN: 972-8394-12-8

INTRODUCTION TO CHARACTERISTIC CLASSES AND INDEX THEORY

Jean-Pierre Schneiders

UNIVERSIDADE DE LISBOA
Faculdade de Ciências
Departamento de Matemática

2000

Author's address:

Laboratoire Analyse, Géométrie et Applications (UMR 7539)
Institut Galilée, Université Paris 13
Avenue Jean-Baptiste Clément
F-93430 Villetaneuse, France

E-mail:

`jps@math.univ-paris13.fr`

URL:

`<http://www-math.univ-paris13.fr/~jps>`

Mathematics Subject Classification (2000): 55R40, 57R20, 19L10, 14C40.

Preface

This book is based on a course given by the author at the university of Lisbon during the academic year 1997–1998. This course was divided in three parts dealing respectively with characteristic classes of real and complex vector bundles, Hirzebruch-Riemann-Roch formula and Atiyah-Singer theorem. In the text which follows, we have decided to treat only the first two subjects. For an introduction to the last one, we refer the reader to [21] for the classical point of view or to [22, 23] for recent developments.

The theory of characteristic classes is a very well developed branch of mathematics and the literature concerning Riemann-Roch theorem is huge. So, we will not try to give a full view of these subjects. We will rather present a few basic but fundamental facts which should help the reader to gain a good idea of the mathematics involved.

Although the reader is assumed to have a good knowledge of homological algebra and topology, we begin with a chapter surveying the results of sheaf theory which are needed in the rest of the book. In particular we recall results concerning acyclic sheaves, taut subspaces, Poincaré-Verdier duality and Borel-Moore homology, illustrating them by means of examples and exercises.

We refer the reader who would like a more detailed treatment of this part to standard texts on sheaf theory (e.g. [11, 6, 28, 18]) and algebraic topology (e.g. [9, 29, 14]). Older works may also be of interest (e.g. [31, 25, 1, 19]).

Chapter 2 is devoted to Euler classes. As a motivation, we begin by proving the classical Lefschetz fixed point formula and applying it to compute the Euler-Poincaré characteristic of a compact oriented topological manifold by means of its Euler class. Next, we study Thom and Euler classes of oriented real vector bundles. In particular, we consider Thom isomorphism, Gysin exact sequence and functorial properties of Euler classes. We end with results on the Euler class of a normal bundle which allow us to link the Euler class of an oriented differential manifold with the one of its tangent bundle.

The first part of Chapter 3 deals mainly with Stiefel-Whitney classes. We define them in the Grothendieck way by means of projective bundles and Euler classes of associated tautological bundles. Then, we establish the pull-back and direct sum formulas. Thanks to the splitting principle, we also obtain a formula for the Stiefel-Whitney classes of a tensor product. In the second part of the chapter, we study in general the characteristic classes of real vector bundles. We begin by classifying real vector bundles of rank r by means of homotopy classes of maps with values in the infinite Grassmannian $G_{\infty,r}$. This establishes a link between the characteristic classes of real vector bundles with coefficients in a group M and the elements of $\mathrm{H}^{\cdot}(G_{\infty,r}; M)$. By computing this last group for $M = \mathbb{Z}_2$, we show that the modulo 2 characteristic classes of real vector bundles are polynomials in the Stiefel-Whitney classes. We end by explaining the usual cohomological classification of real vector bundles and deducing from it that line bundles are classified by their first Stiefel-Whitney class. This allows us to give a criterion for a real vector bundle to be orientable.

Chapter 4 is centered on Chern classes. We begin by adapting most of the results concerning Stiefel-Whitney classes to the complex case. Next, we consider specific results such as the Chern-Weil method of computing Chern classes using the curvature of a connection. We also treat briefly of the Chern character. The last part of the chapter is little bit technical. It is devoted to Iversen's construction of the local Chern character for complexes of complex vector bundles (see [17]) and to its application to the definition of a local Chern character for coherent analytic sheaves.

For more details on the three preceding chapters, the reader may consults classical books on the theory of fiber bundles (e.g. [30, 20, 16]).

The last chapter of this book is about Riemann-Roch theorem. After a short review of the finiteness and duality results for coherent analytic sheaves, we reach the central question of this part *i.e.* how to compute

$$\chi(X; \mathcal{F})$$

for a coherent analytic sheaf on a compact complex analytic manifold X. The answer to this question is essentially due to Hirzebruch (see [15]) and states that

$$\chi(X; \mathcal{F}) = \int_X \operatorname{ch} \mathcal{F} \smile \operatorname{td} TX$$

where $\operatorname{ch} \mathcal{F}$ is the Chern character of $\mathcal{F}$ and $\operatorname{td} TX$ is the Todd class of the tangent bundle of X. To better understand the meaning of this formula, we devote Sections 2–4 to the easy case of line bundles on complex curves. In this situation, X is a compact Riemann surface and we can link the generalized Riemann-Roch theorem considered above with the original results of Riemann and Roch. We end the chapter by proving Hirzebruch-Riemann-Roch theorem for complex projective manifolds. We follow Grothendieck approach (see [5]) by reducing the result to the case of the complex projective space by means of a relative Riemann-Roch formula for embeddings. However, to treat this relative case, we have not followed [5] but used a simpler method based on the ideas of [2, 3, 4] and the proof of the Grothendieck-Riemann-Roch formula in [10].

For bibliographical informations concerning the subject treated in this chapter we refer to [15, 10]. Interesting historical comments may also be found in [7, 8].

It is a pleasure to end this preface by thanking heartily the CMAF for its hospitality during my stay at Lisbon university. I think in particular to T. Monteiro Fernandes who invited me to give the course on which this book is based and suggested to publish it in this collection. I am also grateful to her for taking a set of lectures notes which served as a first draft for this work. All my thanks also to O. Neto and to the various people who attended the course and whose interest has been a strong motivation for turning the raw lecture notes into a book.

Let me finally thank F. Prosmans whose help was invaluable at all the stages of the preparation of the manuscript.

March 2000 Jean-Pierre Schneiders

Contents

1

Survey of sheaf theory

1.1 Abelian presheaves and sheaves

Let X be a topological space and let $\mathcal{O}p(X)$ denote the category of open subsets of X and inclusion maps.

Definition 1.1.1. An *abelian presheaf* on X is a functor $F : \mathcal{O}p(X)^{\mathrm{op}} \to \mathcal{A}b$ where $\mathcal{A}b$ denotes the category of abelian groups. In other words, an abelian presheaf is a law which associates an abelian group $F(U)$ to any open subset U of X and which associates to any open subset $V \subset U$ a *restriction morphism*

$$r^F_{VU} : F(U) \to F(V)$$

in such a way that

$$r^F_{WV} \circ r^F_{VU} = r^F_{WU}$$

for any chain of open subsets $W \subset V \subset U$ of X. We often denote $r^F_{VU}(s)$ simply by $s_{|V}$ when there is no risk of confusion.

A *morphism of abelian presheaves* is simply a morphism of the corresponding functors. More explicitly, a morphism of abelian presheaves $f : F \to G$ is a law which associates to any open subset U of X a morphism

$$f(U) : F(U) \to G(U)$$

in such a way that the diagram

$$\begin{array}{ccc} F(U) & \xrightarrow{f(U)} & G(U) \\ {\scriptstyle r^F_{VU}}\downarrow & & \downarrow{\scriptstyle r^G_{VU}} \\ F(V) & \xrightarrow[f(V)]{} & G(V) \end{array}$$

is commutative for any open subsets $V \subset U$ of X.

With this notion of morphisms, abelian presheaves form an abelian category, denoted $\mathcal{P}sh(X)$.

Examples 1.1.2.

(a) On X, we may consider the abelian presheaf $\mathcal{C}_{0,X}$ defined by setting

$$\mathcal{C}_{0,X}(U) = \{f : U \to \mathbb{C} : f \text{ continuous}\}$$

and defining the restriction morphisms by means of the usual restrictions of functions.

(b) If X is endowed with a Borelian measure μ, we may consider the abelian presheaves $\mu - L_{p,X}$ defined by associating to an open U of X the quotient of the abelian group

$$\{f : U \to \mathbb{C} : \int_U |f|^p dV\mu \leq +\infty\}$$

by the subgroup

$$\{f : U \to \mathbb{C} : f = 0 \ \mu - \text{almost everywhere}\};$$

the restriction morphisms being the obvious ones.

Definition 1.1.3. An *abelian sheaf* on X is an abelian presheaf $\mathcal{F}$ such that

(a) we have

$$\mathcal{F}(\emptyset) = 0;$$

(b) for any open covering $\mathcal{U}$ of an open subset U of X, we have the exact sequence

$$0 \to \mathcal{F}(U) \xrightarrow{\rho_{\mathcal{U}}} \prod_{V \in \mathcal{U}} \mathcal{F}(V) \xrightarrow{\rho'_{\mathcal{U}}} \prod_{V,W \in \mathcal{U}} \mathcal{F}(V \cap W)$$

where

$$\rho_{\mathcal{U}}(s) = (r^{\mathcal{F}}_{VU}(s))_{V \in \mathcal{U}}$$

and

$$\rho'_{\mathcal{U}}((s_V)_{V \in \mathcal{U}}) = (r^{\mathcal{F}}_{(V \cap W)V}(s_V) - r^{\mathcal{F}}_{(V \cap W)W}(s_W))_{V,W \in \mathcal{U}}.$$

A *morphism of abelian sheaves* is a morphism of the underlying abelian presheaves. With this notion of morphisms, abelian sheaves form a full additive subcategory of $\mathcal{P}sh(X)$. We denote it by $\mathcal{S}hv(X)$.

Examples 1.1.4.

(a) Since continuity is a local property, the abelian presheaf $\mathcal{C}_{0,X}$ is clearly an abelian sheaf.

(b) The abelian presheaf $\mu - L_{p,X}$ associated to a Borelian measure μ on X is not in general an abelian sheaf. As a matter of fact, the condition

$$\int_U |f|^p dV\mu < +\infty$$

may be satisfied locally on U without holding globally. Note however that the presheaf $\mu - \mathcal{L}_{p,X}$ of functions which are locally in $\mu - L_{p,X}$ is an abelian sheaf.

(c) If X is a differential manifold, the presheaves $\mathcal{C}^p_{\infty,X}$ and $\mathcal{D}b^p_X$ of smooth and distributional p-forms are clearly abelian sheaves.

(d) Similarly, on a real analytic manifold X, we have the abelian sheaves $\mathcal{A}^p_X$ and $\mathcal{B}^p_X$ of analytic and hyperfunction p-forms.

(e) On a complex analytic manifold X, we have the abelian sheaves $\mathcal{O}_X$, $\mathcal{O}_{\overline{X}}$ of holomorphic and antiholomorphic functions and the abelian sheaves Ω^p_X and $\Omega^p_{\overline{X}}$ of holomorphic and antiholomorphic p-forms. We have also the abelian sheaves $\mathcal{C}^{(p,q)}_{\infty,X}$ and $\mathcal{D}b^{(p,q)}_X$ of smooth and distributional (p,q)-forms and the sheaves $\mathcal{A}^{(p,q)}_X$ and $\mathcal{B}^{(p,q)}_X$ of analytic and hyperfunction (p,q)-forms.

Definition 1.1.5. The *stalk* at $x \in X$ of an abelian presheaf F is the abelian group

$$F_x = \varinjlim_{\substack{U \ni x \\ U \text{ open}}} F(U)$$

where the inductive limit is taken over the set of open neighborhoods of x ordered by $\supset$. We denote

$$r^F_{xU} : F(U) \to F_x$$

the canonical morphism and often use the shorthand notation s_x for

$$r^F_{xU}(s)$$

when there is no risk of confusion.

Remark 1.1.6. Let $x \in X$ and let F be an abelian presheaf on X. To deal with elements of F_x, we only have to know that:

(a) for any $\sigma \in F_x$ there is an open neighborhood U of x in X and $s \in F(U)$ such that $\sigma = s_x$;

(b) if U, U' are two open neighborhoods of x in X and $s \in F(U)$, $s' \in F(U')$ then $s_x = s'_x$ if and only if there is an open neighborhood U'' of x such that $U'' \subset U \cap U'$ and $s_{|U''} = s'_{|U''}$.

Proposition 1.1.7. *Let F be an abelian presheaf on X. Define $\mathcal{A}(F)(U)$ to be the subgroup of $\prod_{x\in U} F_x$ formed by elements σ which are locally in F (i.e. such that for any $x_0 \in U$ there is a neighborhood U_0 of x_0 in U and $s \in F(U_0)$ with $s_x = \sigma_x$ for any $x \in U_0$). Turn $\mathcal{A}(F)$ into an abelian presheaf by setting*

$$\left[r_{VU}^{\mathcal{A}(F)}(\sigma)\right]_x = \sigma_x$$

for any $x \in V$ and consider the morphism

$$a : F \to \mathcal{A}(F)$$

defined by setting

$$[a(U)(s)]_x = s_x$$

for any $x \in U$. Then, $\mathcal{A}(F)$ is an abelian sheaf and for any abelian sheaf $\mathcal{G}$ and any morphism $g : F \to \mathcal{G}$ there is a unique morphism g' making the diagram

$$\begin{tikzcd} F \arrow[r, "a"] \arrow[dr, "g"'] & \mathcal{A}(F) \arrow[d, dashed, "g'"] \\ & \mathcal{G} \end{tikzcd}$$

commutative. Moreover, a induces an isomorphism

$$a_x : F_x \to \mathcal{A}(F)_x$$

for any $x \in X$.

Definition 1.1.8. We call $\mathcal{A}(F)$ the *abelian sheaf associated to F*.

Examples 1.1.9.

(a) The abelian sheaf $\mu - \mathcal{L}_{p,X}$ considered in Examples 1.1.4 is isomorphic to $\mathcal{A}(\mu - L_{p,X})$.

(b) To any abelian group M, we may associate the constant presheaf

$$U \mapsto M.$$

This presheaf is in general not a sheaf. We denote its associated sheaf by M_X and call it the constant sheaf with fiber M. For any open subset U of X, we have

$$M_X(U) = \{\sigma : U \to M \ : \sigma \text{ locally constant}\}.$$

Proposition 1.1.10. *The category $\mathcal{S}hv(X)$ is abelian. The kernel of a morphism*

$$f : \mathcal{F} \to \mathcal{G}$$

is the abelian sheaf

$$U \mapsto \operatorname{Ker} f(U);$$

its cokernel is the abelian sheaf associated to the abelian presheaf

$$U \mapsto \operatorname{Coker} f(U).$$

Proposition 1.1.11. *A sequence of abelian sheaves*

$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \to 0$$

is exact if and only if the sequence of abelian groups

$$0 \to \mathcal{F}_x \xrightarrow{f_x} \mathcal{G}_x \xrightarrow{g_x} \mathcal{H}_x \to 0$$

is exact for any $x \in X$.

Examples 1.1.12.

(a) *Exponential sequence.* Let $\mathcal{C}^*_{0,X}$ denotes the (multiplicative) abelian sheaf formed by non vanishing continuous complex valued functions. Denote

$$\exp : \mathcal{C}_{0,X} \to \mathcal{C}^*_{0,X}$$

the morphism which sends a continuous complex valued function f to $\exp \circ f$ and denote

$$2i\pi : \mathbb{Z}_X \to \mathcal{C}_{0,X}$$

the morphism which sends a locally constant integer valued function n to a complex valued function $2i\pi n$. Then, it follows from the local existence on $\mathbb{C}^*$ of the complex logarithm that

$$0 \to \mathbb{Z}_X \xrightarrow{2i\pi} \mathcal{C}_{0,X} \xrightarrow{\exp} \mathcal{C}^*_{0,X} \to 0$$

is an exact sequence of sheaves.

(b) *de Rham sequences.* Let X be a differential manifold of dimension n and let d denotes the exterior differential. Working by induction on n, it is relatively easy to show that for any convex open subset U of $\mathbb{R}^n$ the sequence

$$0 \to \mathbb{C}_{\mathbb{R}^n}(U) \to \mathcal{C}^0_{\infty,\mathbb{R}^n}(U) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{C}^n_{\infty,\mathbb{R}^n}(U) \to 0$$

is exact. This result, often referred to as the Poincaré lemma, shows directly that

$$0 \to \mathbb{C}_X \to \mathcal{C}^0_{\infty,X} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{C}^n_{\infty,X} \to 0$$

is an exact sequence of abelian sheaves. Similar results hold with $\mathcal{C}_{\infty,X}$ replaced by $\mathcal{D}b_X$ (and by $\mathcal{A}_X$ or $\mathcal{B}_X$ if X is a real analytic manifold).

(c) *Dolbeault sequences.* Let X be a complex analytic manifold. Then, for any smooth (p,q)-form ω we have

$$d\omega = \partial\omega + \overline{\partial}\omega$$

with $\partial\omega$ (resp. $\overline{\partial}\omega$) of type $(p+1,q)$ (resp. $(p,q+1)$). This gives rise to morphisms

$$\partial : \mathcal{C}^{(p,q)}_{\infty,X} \to \mathcal{C}^{(p+1,q)}_{\infty,X}, \qquad \overline{\partial} : \mathcal{C}^{(p,q)}_{\infty,X} \to \mathcal{C}^{(p,q+1)}_{\infty,X}$$

such that $\partial^2 = 0$, $\overline{\partial}^2 = 0$, $\partial\overline{\partial} + \overline{\partial}\partial = 0$. If U is a convex open subset of $\mathbb{C}^n$, one checks by induction on n that the sequences

$$0 \to \Omega^p_{\mathbb{C}^n}(U) \to \mathcal{C}^{(p,0)}_{\infty,\mathbb{C}^n}(U) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{C}^{(p,n)}_{\infty,\mathbb{C}^n}(U) \to 0$$

and

$$0 \to \Omega^p_{\overline{\mathbb{C}^n}}(U) \to \mathcal{C}^{(0,p)}_{\infty,\mathbb{C}^n}(U) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{C}^{(n,p)}_{\infty,\mathbb{C}^n}(U) \to 0$$

are exact. Therefore, we see that

$$0 \to \Omega^p_X \to \mathcal{C}^{(p,0)}_{\infty,X} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{C}^{(p,n)}_{\infty,X} \to 0$$

and

$$0 \to \Omega^p_{\overline{X}} \to \mathcal{C}^{(0,p)}_{\infty,X} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{C}^{(n,p)}_{\infty,X} \to 0$$

are exact sequences of abelian sheaves.

Proposition 1.1.13.

(a) Let $(I_x)_{x\in X}$ be a family of injective abelian groups. Then, the abelian sheaf $\mathcal{I}$ defined by setting

$$\mathcal{I}(U) = \prod_{x\in U} I_x$$

for any open subset U of X and

$$\left[r^{\mathcal{I}}_{VU}(s)\right]_x = s_x$$

for any open subset V of U and any $x \in V$ is injective.

(b) Let $\mathcal{F}$ be an abelian sheaf on X. Then, there is a monomorphism

$$\mathcal{F} \to \mathcal{I}$$

where $\mathcal{I}$ is a sheaf of the type considered in (a). In particular, the abelian category $\mathcal{S}hv(X)$ has enough injective objects.

Remark 1.1.14. As a consequence of the preceding proposition, we get that any functor $F : \mathcal{S}hv(X) \to \mathcal{A}$ where $\mathcal{A}$ is an abelian category has a right derived functor. Note that, in general, $\mathcal{S}hv(X)$ does not have enough projective objects.

1.2 Sections of an abelian sheaf

Definition 1.2.1. Let A be a subset of X and let $\mathcal{F}$ be an abelian sheaf on X.

A *section of $\mathcal{F}$ on A* is an element

$$\sigma \in \prod_{x\in A} \mathcal{F}_x$$

with the property that that for any $x_0 \in A$ there is an open neighborhood U_0 of x_0 in X and $s \in \mathcal{F}(U_0)$ such that

$$\sigma_x = s_x$$

for any $x \in A \cap U_0$. When $A = X$, we call sections of $\mathcal{F}$ on A *global section* of $\mathcal{F}$.

The *support* of a section σ of $\mathcal{F}$ on A is the set

$$\mathrm{supp}(\sigma) = \{x \in A : \sigma_x \neq 0\}.$$

It is the relatively closed subset of A.

Sections of $\mathcal{F}$ on A form an abelian group that we denote by $\Gamma(A;\mathcal{F})$.

Let B be a subset of X such that $B \subset A$ and let $\sigma \in \Gamma(A;\mathcal{F})$. Then, $r^{\mathcal{F}}_{BA}(\sigma)$ is the element of $\Gamma(B;\mathcal{F})$ defined by setting

$$\left[r^{\mathcal{F}}_{BA}(\sigma)\right]_x = \sigma_x$$

for any $x \in B$. We will often use $\sigma_{|B}$ as a shorthand notation for $r^{\mathcal{F}}_{BA}(\sigma)$. Of course, we have

$$\operatorname{supp}(\sigma_{|B}) = \operatorname{supp}(\sigma) \cap B.$$

Remark 1.2.2. Let U be an open subset of X and let $\mathcal{F}$ be an abelian sheaf on X. Then, the canonical morphism

$$\mathcal{F}(U) \to \Gamma(U;\mathcal{F})$$

which sends $s \in \mathcal{F}(U)$ to $(s_x)_{x\in U}$ is an isomorphism. Hereafter, we will often use this isomorphism to identify $\mathcal{F}(U)$ and $\Gamma(U;\mathcal{F})$ without further notice. Note that if V is an open subset of U, the two definitions of $r^{\mathcal{F}}_{VU}$ are compatible with this identification.

Definition 1.2.3. Let X be a topological space and let A be a subspace of X.

We say that A *is relatively Haussdorf in* X if for any $x \neq y$ in A we can find open neighborhoods U and V of x and y in X such that $U \cap V = \emptyset$.

By an *open covering of* A *in* X, we mean a set $\mathcal{U}$ of open subsets of X such that for any $x \in A$ there is $U \in \mathcal{U}$ containing x. Such a covering is *locally finite on* A if any $x \in A$ has a neighborhood V in X for which the set

$$\{U \in \mathcal{U} : U \cap V \neq \emptyset\}$$

is finite.

We say that A is *relatively paracompact in* X if it is relatively Haussdorf and if for any open covering $\mathcal{U}$ of A in X we can find an open covering $\mathcal{V}$ of A in X which is locally finite on A and such that for any $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ with $V \subset U$.

Remark 1.2.4. One checks easily that a subspace A of a topological space X is relatively paracompact if it has a fundamental system of paracompact neighborhoods. This will be the case in particular in the following cases:

(a) X is completely paracompact (e.g. metrizable);

(b) A is closed and X is paracompact.

Other examples of relatively paracompact subspaces are given by relatively Haussdorf compact subspaces.

Proposition 1.2.5. *Let A be a relatively paracompact subspace of X and let $\mathcal{F}$ be an abelian sheaf on X. Then,*

(a) for any $\sigma \in \Gamma(A;\mathcal{F})$ there is an open neighborhood U of A in X and $s \in \Gamma(U;\mathcal{F})$ such that $s_{|A} = \sigma$;

(b) if U, U' are two open neighborhoods of A in X and $s \in \Gamma(U;\mathcal{F})$, $s' \in \Gamma(U',\mathcal{F})$ then $s_{|A} = s'_{|A}$ if and only if there is an open neighborhood U'' of A such that $U'' \subset U \cap U'$ and $s_{|U''} = s'_{|U''}$.

In other words, we have

$$\Gamma(A;\mathcal{F}) = \varinjlim_{\substack{U \supset A \\ U \text{ open}}} \Gamma(U;\mathcal{F}).$$

Proof. See [24]. ☐

Proposition 1.2.6. *Let A be a topological subspace of X and let $\mathcal{F}$ be an abelian sheaf on X. Then, the abelian presheaf $\mathcal{F}_{|A}$ defined by setting*

$$\mathcal{F}_{|A}(U) = \Gamma(U;\mathcal{F})$$

for any open subset U of A and

$$r^{\mathcal{F}_{|A}}_{VU} = r^{\mathcal{F}}_{VU}$$

for any chain $V \subset U$ of open subsets of A is an abelian sheaf.

Remark 1.2.7. It follows from the preceding results that sections of $\mathcal{F}$ on A may be considered as the global sections of the abelian sheaf $\mathcal{F}_{|A}$.

1.3 Cohomology with supports

Definition 1.3.1. A *family of supports* of X is a set Φ of closed subsets of X such that

(a) if F_1 is a closed subset of X included in $F_2 \in \Phi$, then $F_1 \in \Phi$;

(b) for any F_1, $F_2 \in \Phi$, there is $F_3 \in \Phi$ such that $F_1 \cup F_2 \subset F_3$.

Let $\mathcal{F}$ be an abelian sheaf on X. Then global sections s of $\mathcal{F}$ such that

$$\operatorname{supp}(s) \in \Phi$$

form an abelian group that we denote $\Gamma_\Phi(X;\mathcal{F})$.

Examples 1.3.2.

(a) The set Φ_X of all the closed subsets of X is clearly a family of supports and we have
$$\Gamma_{\Phi_X}(X;\mathcal{F}) = \Gamma(X;\mathcal{F})$$
for any abelian sheaf $\mathcal{F}$.

(b) Let F be a closed subset of X. Then, the set Φ_F of all the closed subsets of F is a family of supports. In this case we set for short
$$\Gamma_F(X;\mathcal{F}) = \Gamma_{\Phi_F}(X;\mathcal{F})$$
for any abelian sheaf $\mathcal{F}$. Note that this special case allows us to recover the general one. As a matter of fact, we have
$$\Gamma_\Phi(X;\mathcal{F}) = \varinjlim_{F\in\Phi} \Gamma_F(X;\mathcal{F}).$$

(b) Let X be a Haussdorf space. Then, the set Φ_c of all compact subsets of X is a family of supports. We set for short
$$\Gamma_c(X;\mathcal{F}) = \Gamma_{\Phi_c}(X;\mathcal{F})$$
for any abelian sheaf $\mathcal{F}$.

Let Φ be a family of supports of X.

Proposition 1.3.3. *The functor*
$$\Gamma_\Phi(X;\cdot) : \mathcal{S}hv(X) \to \mathcal{A}b$$
is left exact and has a right derived functor
$$\mathrm{R}\Gamma_\Phi(X;\cdot) : \mathcal{D}^+(\mathcal{S}hv(X)) \to \mathcal{D}^+(\mathcal{A}b).$$

Proof. The left exactness follows directly from the structure of kernels in $\mathcal{S}hv(X)$. The existence of the right derived functor follows from Remark 1.1.14. □

Definition 1.3.4. Let $\mathcal{F}$ be an abelian sheaf on X. We define the k-th *cohomology group of X with coefficients in $\mathcal{F}$ and supports in* Φ as the group
$$\mathrm{H}^k[\mathrm{R}\Gamma_\Phi(X;\mathcal{F})].$$
To lighten notations, we denote it by
$$\mathrm{H}^k_\Phi(X;\mathcal{F}).$$

If Φ is the family of all closed subsets of X, we shorten the notation by dropping the Φ. Similarly, if Φ is the family Φ_F (resp. Φ_c) considered in Examples 1.3.2, we replace it by F (resp. c).

An abelian sheaf $\mathcal{F}$ is Φ-*acyclic* if $\mathrm{H}^k_\Phi(X;\mathcal{F}) = 0$ for any $k > 0$.

Remark 1.3.5. Let $\mathcal{F}$ be an abelian sheaf on X. By a well-known result of homological algebra

$$\mathrm{R}\Gamma_\Phi(X;\mathcal{F}) \simeq \Gamma_\Phi(X;\mathcal{R}^\cdot)$$

if $\mathcal{R}^\cdot$ is a right resolution of $\mathcal{F}$ by Φ-acyclic sheaves. The aim of the next section is to give basic examples of such sheaves

1.4 Flabby and soft abelian sheaves

Definition 1.4.1. An abelian sheaf $\mathcal{F}$ on X is *flabby* if for any chain $U_1 \subset U_2$ of open subsets of X

$$r^{\mathcal{F}}_{U_1U_2} : \Gamma(U_2;\mathcal{F}) \to \Gamma(U_1;\mathcal{F})$$

is an epimorphism.

Examples 1.4.2.

(a) Any injective abelian sheaf is flabby.

(b) Let $(M_x)_{x\in X}$ be a family of abelian groups. Then,

$$U \mapsto \prod_{x\in U} M_x$$

is a flabby sheaf.

(c) The sheaf $\mu - \mathcal{L}_{p,X}$ of Examples 1.1.4 is flabby.

(d) The sheaf $\mathcal{B}_X$ of hyperfunctions is flabby.

Proposition 1.4.3. *A flabby abelian sheaf $\mathcal{F}$ is Φ-acyclic for any family of supports Φ. Moreover, if in an exact sequence of abelian sheaves*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

$\mathcal{F}$ and $\mathcal{G}$ are flabby, then so is $\mathcal{H}$.

Definition 1.4.4. An abelian sheaf $\mathcal{F}$ on X is Φ-*soft* if for any chain $F_1 \subset F_2$ of Φ

$$r^{\mathcal{F}}_{F_1F_2} : \Gamma(F_2;\mathcal{F}) \to \Gamma(F_1;\mathcal{F})$$

is an epimorphism.

Examples 1.4.5.

(a) Assume X is normal. Thanks to Urysohn's extension result, it is clear that the abelian sheaf $\mathcal{C}_{0,X}$ is Φ-soft for any family of supports Φ.

(b) If X is a differential manifold, it follows from the existence of partitions of unity that the abelian sheaves $\mathcal{C}^p_{\infty,X}$ and $\mathcal{D}b^p_X$ are Φ-soft for any family of supports Φ. The same is true of the sheaves $\mathcal{C}^{(p,q)}_{\infty,X}$ and $\mathcal{D}b^{(p,q)}_X$ if X is a complex analytic manifold.

Definition 1.4.6. A family of supports Φ is *paracompactifying* if

(a) any $F \in \Phi$ is paracompact;

(b) for any $F_1 \in \Phi$, there is $F_2 \in \Phi$ with $F_1 \subset \overset{\circ}{F_2}$.

Examples 1.4.7.

(a) If X is paracompact, then the family formed by the closed subsets of X is paracompactifying.

(b) If X is a locally compact space, then the family formed by the compact subsets of X is paracompactifying.

Proposition 1.4.8. *Assume the abelian sheaf $\mathcal{F}$ is Φ-soft and the family Φ is paracompactifying. Then, $\mathcal{F}$ is Φ-acyclic. Moreover, if in the exact sequence of abelian sheaves*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

$\mathcal{F}$ and $\mathcal{G}$ are Φ-soft, then so is $\mathcal{H}$.

Corollary 1.4.9 (de Rham and Dolbeault theorems).

(a) For any differential manifold X, we have the canonical isomorphisms

$$\mathrm{H}^k(X;\mathbb{C}_X) \simeq \mathrm{H}^k(\Gamma(X;\mathcal{C}^{\cdot}_{\infty,X})) \simeq \mathrm{H}^k(\Gamma(X;\mathcal{D}b^{\cdot}_X))$$

and

$$\mathrm{H}^k_c(X;\mathbb{C}_X) \simeq \mathrm{H}^k(\Gamma_c(X;\mathcal{C}^{\cdot}_{\infty,X})) \simeq \mathrm{H}^k(\Gamma_c(X;\mathcal{D}b^{\cdot}_X))$$

for any $k \in \mathbb{N}$.

(b) For any complex analytic manifold X, we have the canonical isomorphisms

$$\mathrm{H}^k(X;\Omega^p_X) \simeq \mathrm{H}^k(\Gamma(X;\mathcal{C}^{(p,\cdot)}_{\infty,X})) \simeq \mathrm{H}^k(\Gamma(X;\mathcal{D}b^{(p,\cdot)}_X))$$

and

$$\mathrm{H}^k_c(X;\Omega^p_X) \simeq \mathrm{H}^k(\Gamma_c(X;\mathcal{C}^{(p,\cdot)}_{\infty,X})) \simeq \mathrm{H}^k(\Gamma_c(X;\mathcal{D}b^{(p,\cdot)}_X))$$

for any k, $p \in \mathbb{N}$.

Proof. Thanks to Examples 1.4.5 (b) and Examples 1.4.7, this follows directly from Remark 1.3.5 and Proposition 1.4.8. □

Exercise 1.4.10. *Let B_n denote the open unit ball of $\mathbb{R}^n$. Show that for $n \geq 1$*

$$\mathrm{H}^k(B_n; \mathbb{C}_{B_n}) \simeq \begin{cases} \mathbb{C} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Solution. Recall that for any convex open subset U of $\mathbb{R}^n$, the Poincaré lemma for d shows that the sequence

$$0 \to \Gamma(U; \mathbb{C}_{\mathbb{R}^n}) \to \Gamma(U; \mathcal{C}^0_{\infty,\mathbb{R}^n}) \to \cdots \to \Gamma(U; \mathcal{C}^n_{\infty,\mathbb{R}^n}) \to 0$$

is exact. Therefore, by the de Rham theorem we have

$$\mathrm{H}^k(U; \mathbb{C}_U) \simeq \begin{cases} \mathbb{C} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Since B_n is convex, the conclusion follows. □

1.5 Cohomology of subspaces and tautness

Definition 1.5.1. Let X be a topological space and let Φ be a family of supports on X. We say that a subspace A of X is Φ-*taut* if the canonical morphism

$$\varinjlim_{\substack{U \supset A \\ U \text{ open}}} \mathrm{H}^k_{\Phi \cap U}(U; \mathcal{F}_{|U}) \to \mathrm{H}^k_{\Phi \cap A}(A; \mathcal{F}_{|A})$$

is an isomorphism for any $k \geq 0$ and any abelian sheaf $\mathcal{F}$ on X.

Remark 1.5.2. It is easily seen that a subspace A of X is Φ-*taut* if and only if for any flabby sheaf $\mathcal{F}$ on X

(a) the abelian sheaf $\mathcal{F}_{|A}$ is $\Phi \cap A$-acyclic;

(b) the canonical morphism

$$\Gamma_\Phi(X; \mathcal{F}) \to \Gamma_{\Phi \cap A}(A; \mathcal{F}_{|A})$$

is surjective.

In this case, it follows that

$$\mathcal{F} \mapsto \mathrm{R}\Gamma(A; \mathcal{F}_{|A})$$

is the right derived functor of

$$\mathcal{F} \mapsto \Gamma(A; \mathcal{F}).$$

One should however be aware that this result is false in general.

Proposition 1.5.3. *Let X be a topological space and let Φ be a family of supports on X. Assume A is a topological subspace of X. Then A is Φ-taut in the following cases:*

(a) Φ is arbitrary and A is open;

(b) Φ is paracompactifying and A is closed;

(c) Φ is paracompactifying and X is completely paracompact;

(d) Φ is the family of all closed subset X and A is both compact and relatively Haussdorf.

Exercise 1.5.4. *Let $\overline{B}_n$ denote the closed unit ball of $\mathbb{R}^n$. Show that for $n \geq 1$*

$$\mathrm{H}^k(\overline{B}_n; \mathbb{C}_{\overline{B}_n}) \simeq \begin{cases} \mathbb{C} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Solution. Since $\overline{B}_n$ has a fundamental system of neighborhoods formed by open balls of $\mathbb{R}^n$, this follows directly from Proposition 1.5.3 and Exercise 1.4.10 □

1.6 Excision and Mayer-Vietoris sequences

Definition 1.6.1. Let A be a subset of X and let Φ be a family of supports of X. We set

$$\Phi_{|A} = \{F \in \Phi : F \subset A\}$$

and

$$\Phi \cap A = \{F \cap A : F \in \Phi\}.$$

Remark 1.6.2. Let A be a subset of X and let Φ a family of supports of X. Clearly, for any abelian sheaf $\mathcal{F}$ on X, we have canonical morphisms

$$\Gamma_{\Phi_{|A}}(X; \mathcal{F}) \to \Gamma_{\Phi}(X; \mathcal{F})$$

and

$$\Gamma_{\Phi}(X; \mathcal{F}) \to \Gamma_{\Phi \cap A}(A; \mathcal{F})$$

which induce similar morphisms at the level of derived functors.

Proposition 1.6.3 (Excision). *Let A be a subset of X and let Φ be a family of supports of X. Assume either A is open or A is closed and Φ is paracompactifying. Then, for any object $\mathcal{F}^\cdot$ of $\mathcal{D}^+(\mathcal{S}hv(X))$, we have the canonical distinguished triangle*

$$\mathrm{R}\Gamma_{\Phi_{|X\setminus A}}(X;\mathcal{F}^\cdot) \to \mathrm{R}\Gamma_\Phi(X;\mathcal{F}^\cdot) \to \mathrm{R}\Gamma_{\Phi\cap A}(A;\mathcal{F}^\cdot) \xrightarrow{+1} .$$

In particular, for any abelian sheaf $\mathcal{F}$, we have the excision long exact sequence:

$$\begin{array}{l} \cdots\cdots \mathrm{H}^k_{\Phi_{|X\setminus A}}(X;\mathcal{F}) \longrightarrow \mathrm{H}^k_\Phi(X;\mathcal{F}) \longrightarrow \mathrm{H}^k_{\Phi\cap A}(A;\mathcal{F}) \longrightarrow \\ \hookrightarrow \mathrm{H}^{k+1}_{\Phi_{|X\setminus A}}(X;\mathcal{F}) \to \mathrm{H}^{k+1}_\Phi(X;\mathcal{F}) \to \mathrm{H}^{k+1}_{\Phi\cap A}(A;\mathcal{F}) \cdots\cdots \end{array}$$

Proof. Let us recall the proof of this result since a detailed understanding of its mechanism will be necessary in various parts of this book. We treat only the case where A is open; the other case being similar.

Assume $\mathcal{G}$ is a flabby sheaf and let $s \in \Gamma(A;\mathcal{G})$ be such that

$$\mathrm{supp}(s) \subset F \cap A$$

with F in Φ. The zero section on $X \setminus F$ and the section s on A coincide on $(X \setminus F) \cap A = A \setminus F$. Therefore, there is a section s' of $\mathcal{G}$ on $(X \setminus F) \cup A$ such that $s'_{|X\setminus F} = 0$, $s'_{|A} = s$. Since $\mathcal{G}$ is flabby and $(X \setminus F) \cup A$ is open, we may extend s' into a section s'' of $\mathcal{G}$ on X. For this section, we have

$$s''_{|X\setminus F} = 0, \qquad s''_{|A} = s.$$

Hence, $\mathrm{supp}\, s'' \subset F$ and belongs to Φ. These considerations show that

$$\Gamma_\Phi(X;\mathcal{G}) \to \Gamma_{\Phi\cap A}(A;\mathcal{G})$$

is an epimorphism. Moreover, a simple computation shows that

$$0 \to \Gamma_{\Phi_{|X\setminus A}}(X;\mathcal{F}) \to \Gamma_\Phi(X;\mathcal{F}) \to \Gamma_{\Phi\cap A}(A;\mathcal{F})$$

is exact for any abelian sheaf $\mathcal{F}$. It follows that if $\mathcal{G}^\cdot$ is a flabby resolution of the complex $\mathcal{F}^\cdot$, then

$$0 \to \Gamma_{\Phi_{|X\setminus A}}(X;\mathcal{G}^\cdot) \to \Gamma_\Phi(X;\mathcal{G}^\cdot) \to \Gamma_{\Phi\cap A}(A;\mathcal{G}^\cdot) \to 0$$

is an exact sequence of complexes of abelian groups. Since flabby sheaves are acyclic for the various functors involved, we get the announced distinguished triangle. The last part of the result follows from the snakes' lemma. $\square$

Remark 1.6.4. Let us recall that the snakes' lemma states that an exact sequence of complexes of abelian groups

$$0 \to A^{\cdot} \xrightarrow{u^{\cdot}} B^{\cdot} \xrightarrow{v^{\cdot}} C^{\cdot} \to 0$$

induces a long exact sequence of cohomology

$$\begin{tikzcd} \cdots \mathrm{H}^k(A^{\cdot}) \arrow[r, "\mathrm{H}^k(u^{\cdot})"] & \mathrm{H}^k(B^{\cdot}) \arrow[r, "\mathrm{H}^k(v^{\cdot})"] & \mathrm{H}^k(C^{\cdot}) \arrow[dll, "\delta^k"] \\ \mathrm{H}^{k+1}(A^{\cdot}) \arrow[r, "\mathrm{H}^{k+1}(u^{\cdot})"] & \mathrm{H}^{k+1}(B^{\cdot}) \arrow[r, "\mathrm{H}^{k+1}(v^{\cdot})"] & \mathrm{H}^{k+1}(C^{\cdot}) \cdots \end{tikzcd}$$

where δ^k is defined as follows. Let $[c^k]$ be a cohomology class in $\mathrm{H}^k(C^{\cdot})$. Since v^k is surjective, there is $b^k \in B^k$ such that

$$v^k(b^k) = c^k.$$

Using the fact that $d^k(c^k) = 0$, one sees that $d^k(b^k)$ is in $\operatorname{Ker} v^{k+1}$. Hence, there is $a^{k+1} \in A^{k+1}$ such that $u^{k+1}(a^{k+1}) = d^k(b^k)$. The cohomology class of a^{k+1} is the image of $[c^k]$ by δ^k. A way to remember this definition is to follow the dotted path in the following diagram:

$$\begin{tikzcd} A^k \arrow[r, "u^k"] \arrow[d] & B^k \arrow[r, "v^k"] \arrow[d] \arrow[d, dashed, bend right] & C^k \arrow[d] \arrow[l, dashed, bend left] \\ A^{k+1} \arrow[r, "u^{k+1}"] & B^{k+1} \arrow[r, "v^{k+1}"] \arrow[l, dashed, bend left] & C^{k+1} \end{tikzcd}$$

Proposition 1.6.5 (Mayer-Vietoris). *Let A, B be two subsets of X and let Φ be a family of supports on X. Assume that either A and B are open or A and B are closed and Φ is paracompactifying. Then, for any object $\mathcal{F}^{\cdot}$ of $\mathcal{D}^+(\mathcal{S}hv(X))$, we have the canonical distinguished triangle*

$$\mathrm{R}\Gamma_{\Phi\cap(A\cup B)}(A \cup B; \mathcal{F}^{\cdot}) \to \mathrm{R}\Gamma_{\Phi\cap A}(A; \mathcal{F}^{\cdot}) \oplus \mathrm{R}\Gamma_{\Phi\cap B}(B; \mathcal{F}^{\cdot}) \to \mathrm{R}\Gamma_{\Phi\cap(A\cap B)}(A \cap B; \mathcal{F}^{\cdot}) \xrightarrow{+1}$$

In particular, if $\mathcal{F}$ is an abelian sheaf, we have the Mayer-Vietoris long exact sequence

$$\begin{tikzcd} \cdots \mathrm{H}^k_{\Phi\cap(A\cup B)}(A\cup B;\mathcal{F}) \arrow[r] & \mathrm{H}^k_{\Phi\cap A}(A;\mathcal{F}) \oplus \mathrm{H}^k_{\Phi\cap B}(B;\mathcal{F}) \arrow[r] & \mathrm{H}^k_{\Phi\cap(A\cap B)}(A\cap B;\mathcal{F}) \arrow[dll] \\ \mathrm{H}^{k+1}_{\Phi\cap(A\cup B)}(A\cup B;\mathcal{F}) \arrow[r] & \mathrm{H}^{k+1}_{\Phi\cap A}(A;\mathcal{F}) \oplus \mathrm{H}^{k+1}_{\Phi\cap B}(B;\mathcal{F}) \arrow[r] & \mathrm{H}^{k+1}_{\Phi\cap(A\cap B)}(A\cap B;\mathcal{F}) \cdots \end{tikzcd}$$

Proof. The proof being similar to that of Proposition 1.6.3, we will not recall it in details. We only recall that it is based on the fact that the sequence

$$0 \to \Gamma_{\Phi\cap(A\cup B)}(A\cup B;\mathcal{G}) \xrightarrow{\alpha} \Gamma_{\Phi\cap A}(A;\mathcal{G}) \oplus \Gamma_{\Phi\cap B}(B;\mathcal{G}) \xrightarrow{\beta} \Gamma_{\Phi\cap(A\cap B)}(A\cap B;\mathcal{G}) \to 0$$

where $\alpha(s) = (s_{|A}, s_{|A})$, $\beta(s,s') = s_{|A\cap B} - s'_{|A\cap B}$ is exact when A, B are open and $\mathcal{G}$ is flabby or when A, B are closed, Φ is paracompactifying and $\mathcal{G}$ is Φ-soft. $\square$

Exercise 1.6.6.

(a) *Let S_n denotes the unit sphere in $\mathbb{R}^{n+1}$. By using Mayer-Vietoris sequence and de Rham theorem, show that for $n \geq 1$*

$$\mathrm{H}^k(S_n;\mathbb{C}_{S_n}) \simeq \begin{cases} \mathbb{C} & \text{if } k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

(b) *Show by excision that for $n \geq 1$*

$$\mathrm{H}^k_c(B_n;\mathbb{C}_{B_n}) \simeq \begin{cases} \mathbb{C} & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

Solution. (a) Assume $n \geq 1$. Set

$$S_n^+ = \{x \in S_n : x_{n+1} \geq 0\}, \quad S_n^- = \{x \in S_n : x_{n+1} \leq 0\}$$

and identify S_{n-1} with $S_n^+ \cap S_n^-$ (see figure 1.6.1). Since $S_n = S_n^+ \cup S_n^-$, we have the Mayer-Vietoris long exact sequence

$$\cdots\cdots \mathrm{H}^k(S_n;\mathbb{C}) \longrightarrow \mathrm{H}^k(S_n^+;\mathbb{C}) \oplus \mathrm{H}^k(S_n^-;\mathbb{C}) \longrightarrow \mathrm{H}^k(S_{n-1};\mathbb{C}) \longrightarrow$$
$$\longrightarrow \mathrm{H}^{k+1}(S_n;\mathbb{C}) \to \mathrm{H}^{k+1}(S_n^+;\mathbb{C}) \oplus \mathrm{H}^{k+1}(S_n^-;\mathbb{C}) \to \mathrm{H}^{k+1}(S_{n-1};\mathbb{C}) \cdots\cdots$$

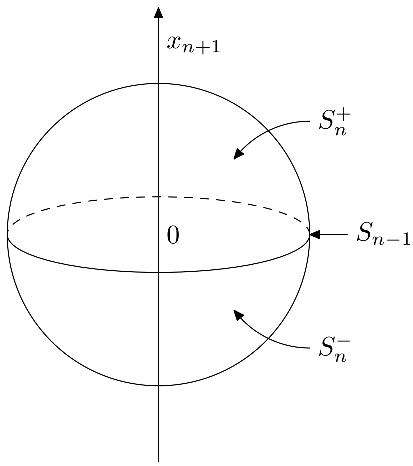
Recall that, for any $\epsilon \in]0,1[$, $\{x \in S_n : x_{n+1} > -\epsilon\}$ is an open subset of S_n which is diffeomorphic to a ball of $\mathbb{R}^n$. Therefore, working as in (a), we see that

$$\mathrm{H}^k(S_n^+;\mathbb{C}) \simeq \begin{cases} \mathbb{C} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Of course, there is a similar result for S_n^-. It follows that

$$\mathrm{H}^{k+1}(S_n;\mathbb{C}) \simeq \mathrm{H}^k(S_{n-1};\mathbb{C}) \tag{*}$$

Figure 1.6.1:



if $k > 0$ and that $\mathrm{H}^0(S_n;\mathbb{C})$ and $\mathrm{H}^1(S_n;\mathbb{C})$ are isomorphic to the kernel and the cokernel of the morphism

$$\begin{aligned} \mathrm{H}^0(S_n^+;\mathbb{C}) \oplus \mathrm{H}^0(S_n^-;\mathbb{C}) &\to \mathrm{H}^0(S_{n-1};\mathbb{C}) \\ (\varphi,\psi) &\mapsto \varphi_{|S_{n-1}} - \psi_{|S_{n-1}} \end{aligned} \tag{**}$$

Note that since S_n^+ and S_n^- are clearly connected spaces, the locally constant complex valued functions φ and ψ are in fact constant.

Let us assume first that $n = 1$. Since $S_0 = \{0, 1\}$, we have

$$\mathrm{H}^k(S_0;\mathbb{C}) \simeq \begin{cases} \mathbb{C}^2 & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the morphism (**) becomes up to isomorphisms the additive map

$$\begin{aligned} \mathbb{C}^2 &\to \mathbb{C}^2 \\ (x, y) &\mapsto (x - y, x - y) \end{aligned}$$

It follows that $\mathrm{H}^1(S_1;\mathbb{C}) \simeq \mathbb{C}$ and that $\mathrm{H}^0(S_1;\mathbb{C}) \simeq \mathbb{C}$. This last isomorphism reflecting the fact that the circle S_1 is connected.

Assume now that $n > 1$ and that

$$\mathrm{H}^k(S_{n-1};\mathbb{C}) \simeq \begin{cases} \mathbb{C} & \text{if } k = 0, n-1 \\ 0 & \text{otherwise} \end{cases}$$

The morphism (**) now becomes equivalent to the additive map

$$\begin{aligned} \mathbb{C}^2 &\to \mathbb{C} \\ (x, y) &\mapsto x - y \end{aligned}$$

Hence, $\mathrm{H}^1(S_n; \mathbb{C}) \simeq 0$ and $\mathrm{H}^0(S_n; \mathbb{C}) \simeq \mathbb{C}$. Moreover, thanks to (*) we see that for $k > 1$ we have

$$\mathrm{H}^k(S_n; \mathbb{C}) \simeq \begin{cases} \mathbb{C} & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

The conclusion follows by induction.

(b) Assume $n \geq 1$. Since $\overline{B}_n \setminus B_n = S_{n-1}$ and since any closed subset of $\overline{B}_n$ is compact, we have the excision distinguished triangle

$$\mathrm{R}\Gamma_c(B_n; \mathbb{C}) \to \mathrm{R}\Gamma(\overline{B}_n; \mathbb{C}) \to \mathrm{R}\Gamma(S_{n-1}; \mathbb{C}) \xrightarrow{+1}$$

From the associated long exact sequence, we deduce that

$$\mathrm{H}^k(S_{n-1}; \mathbb{C}) \simeq \mathrm{H}_c^{k+1}(B_n; \mathbb{C})$$

for $k > 0$ since in this case

$$\mathrm{H}^k(\overline{B}_n; \mathbb{C}) \simeq \mathrm{H}^{k+1}(\overline{B}_n; \mathbb{C}) \simeq 0.$$

Moreover, we see also that $\mathrm{H}_c^0(B_n; \mathbb{C})$ and $\mathrm{H}_c^1(B_n; \mathbb{C})$ are the kernel and cokernel of the morphism

$$\begin{aligned} \mathrm{H}^0(\overline{B}_n; \mathbb{C}) &\to \mathrm{H}^0(S_{n-1}; \mathbb{C}) \\ \varphi &\mapsto \varphi_{|S_{n-1}} \end{aligned}$$

For $n = 1$, this morphism is equivalent to

$$\begin{aligned} \mathbb{C} &\to \mathbb{C}^2 \\ x &\mapsto (x, x) \end{aligned}$$

and we get $\mathrm{H}_c^0(B_1; \mathbb{C}) \simeq 0$ and $\mathrm{H}_c^1(B_1; \mathbb{C}) \simeq \mathbb{C}$.

For $n > 1$, it becomes equivalent to

$$\begin{aligned} \mathbb{C} &\to \mathbb{C} \\ x &\mapsto x \end{aligned}$$

and we get $\mathrm{H}_c^0(B_n; \mathbb{C}) \simeq \mathrm{H}_c^1(B_n; \mathbb{C}) \simeq 0$. The conclusion follows by induction on n. $\square$

Exercise 1.6.7. *Let $I = [a, b]$ be a compact interval of $\mathbb{R}$. Show by using tautness and a suitable Mayer-Vietoris sequence that for any abelian sheaf $\mathcal{F}$ on I, we have*

$$\mathrm{H}^k(I; \mathcal{F}) = 0$$

for $k > 1$. Establish also that this relation holds for $k = 1$ if

$$\Gamma(I; \mathcal{F}) \to \mathcal{F}_x$$

is an epimorphism for every $x \in I$. As an application compute

$$\mathrm{H}^{\cdot}(I; M_I)$$

for any abelian group M.

Solution. Fix $k > 0$ and assume there is $c \in \mathrm{H}^k(I; \mathcal{F})$ which is non-zero. Set

$$x_0 = \inf\{x \in I : c_{|[a,x]} \neq 0\}.$$

Since by tautness

$$\varinjlim_{x > x_0} \mathrm{H}^k([a, x]; \mathcal{F}) \simeq \mathrm{H}^k([a, x_0]; \mathcal{F})$$

we see that $c_{|[a,x_0]} \neq 0$ and hence that $x_0 > a$. Since

$$\varinjlim_{x < x_0} \mathrm{H}^k([x, x_0]; \mathcal{F}) = \mathrm{H}^k(\{x_0\}; \mathcal{F}) = 0,$$

there is $x < x_0$ with $c_{|[x,x_0]} = 0$. For such an x, the decomposition

$$[a, x_0] = [a, x] \cup [x, x_0]$$

gives the Mayer-Vietoris exact sequence

$$\cdots\cdots \mathrm{H}^{k-1}([a, x_0]; \mathcal{F}) \longrightarrow \mathrm{H}^{k-1}([a, x]; \mathcal{F}) \oplus \mathrm{H}^{k-1}([x, x_0]; \mathcal{F}) \longrightarrow \mathrm{H}^{k-1}(\{x\}; \mathcal{F}) \longrightarrow$$

$$\longrightarrow \mathrm{H}^{k}([a, x_0]; \mathcal{F}) \longrightarrow \mathrm{H}^{k}([a, x]; \mathcal{F}) \oplus \mathrm{H}^{k}([x, x_0]; \mathcal{F}) \longrightarrow \mathrm{H}^{k}(\{x\}; \mathcal{F}) \cdots\cdots$$

If $k > 1$ or $k = 1$ and

$$\Gamma(I; \mathcal{F}) \to \mathcal{F}_x$$

is an epimorphism, it follows from this sequence that

$$\mathrm{H}^k([a, x_0]; \mathcal{F}) \xrightarrow{\sim} \mathrm{H}^k([a, x]; \mathcal{F}) \oplus \mathrm{H}^k([x, x_0]; \mathcal{F}).$$

This gives us a contradiction since both $c_{|[a,x]}$ and $c_{|[x,x_0]}$ are 0 although $c_{|[a,x_0]} \neq 0$.

The application to $\mathcal{F} = M_I$ is obvious. We get

$$\mathrm{H}^k(I; M_I) = \begin{cases} M & \text{if } k = 0; \\ 0 & \text{otherwise.} \end{cases}$$

□

1.7 Inverse and direct images

Definition 1.7.1. Let $f : X \to Y$ be a morphism of topological spaces and let $\mathcal{G}$ be an abelian sheaf on Y.

Let U be an open subset of X. We define $f^{-1}(\mathcal{G})(U)$ as the abelian subgroup of $\prod_{x\in U} \mathcal{G}_{f(x)}$ formed by the elements σ such that for any $x_0 \in U$ there is an open neighborhood U_0 of x_0 in U, an open neighborhood V_0 of $f(U_0)$ and $s \in \mathcal{G}(V_0)$ such that $\sigma_x = s_{f(x)}$ for any $x \in U_0$. Clearly,

$$U \mapsto f^{-1}(\mathcal{G})(U)$$

is an abelian sheaf on X. We call it the *inverse image* of $\mathcal{G}$ by f. Note that by construction there is a canonical pull-back morphism

$$f^* : \Gamma(Y; \mathcal{G}) \to \Gamma(X; f^{-1}(\mathcal{G})).$$

Remark 1.7.2. It follows at once from the preceding definition that we may identify $f^{-1}(\mathcal{G})_x$ and $\mathcal{G}_{f(x)}$. In particular, the functor

$$f^{-1} : \mathcal{S}hv(Y) \to \mathcal{S}hv(X)$$

is exact.

Proposition 1.7.3. *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of topological spaces. Then,*

$$f^{-1}(g^{-1}(\mathcal{H})) \simeq (g \circ f)^{-1}(\mathcal{H})$$

canonically and functorially for $\mathcal{H} \in \mathcal{S}hv(Z)$. Moreover,

$$(\mathrm{id}_X^{-1})(\mathcal{F}) \simeq \mathcal{F}$$

canonically and functorially for $\mathcal{F} \in \mathcal{S}hv(X)$.

Examples 1.7.4.

(a) Let $a_X : X \to \{\mathrm{pt}\}$ be the morphism which maps the topological space X to a point. Identifying sheaves on $\{\mathrm{pt}\}$ with their global sections, we have

$$M_X \simeq a_X^{-1}(M)$$

for any abelian group M.

(b) Combining the preceding proposition with example (a), we get a canonical isomorphism

$$f^{-1}(M_Y) \simeq M_X$$

for any abelian group M and any morphism of topological spaces $f : X \to Y$.

(c) Let $i : A \to X$ be the canonical inclusion of a subspace A of X and let $\mathcal{F}$ be an abelian sheaf on X. One checks easily that

$$i^{-1}(\mathcal{F}) = \mathcal{F}_{|A}.$$

Definition 1.7.5. We say that $(X;\mathcal{F})$ is an *abelian sheafed space* if X is a topological space and $\mathcal{F}$ is an abelian sheaf on X. We say that

$$(f;\varphi) : (X;\mathcal{F}) \to (Y;\mathcal{G})$$

is a *morphism of abelian sheafed spaces* if $(X;\mathcal{F})$ and $(Y;\mathcal{G})$ are abelian sheafed spaces, $f : X \to Y$ is a morphism of topological spaces and

$$\varphi : f^{-1}\mathcal{G} \to \mathcal{F}$$

is a morphism of sheaves.

Examples 1.7.6.

(a) Let $f : X \to Y$ be a morphism of topological spaces. Then the canonical isomorphism

$$f^* : f^{-1}M_Y \to M_X$$

gives rise to a morphism of sheafed spaces

$$(f;f^*) : (X;M_X) \to (Y;M_Y).$$

(b) Let $f : X \to Y$ be a morphism of topological spaces. Then,

$$g \mapsto g \circ f$$

induces a canonical morphism of abelian sheaves

$$f^* : f^{-1}\mathcal{C}_{0,Y} \to \mathcal{C}_{0,X}$$

and hence a morphism of sheafed spaces

$$(f;f^*) : (X;\mathcal{C}_{0,X}) \to (Y;\mathcal{C}_{0,Y}).$$

(c) Similarly, a morphism of differential manifolds $f : X \to Y$ induces morphisms of abelian sheaves

$$f^* : f^{-1}\mathcal{C}^p_{\infty,Y} \to \mathcal{C}^p_{\infty,X}$$

corresponding to the pull-back of differential p-forms. Hence, we have a canonical morphism of sheafed spaces

$$(f;f^*) : (X;\mathcal{C}^p_{\infty,X}) \to (Y;\mathcal{C}^p_{\infty,Y}).$$

Proposition 1.7.7. *A morphism of sheafed spaces*

$$(f;\varphi) : (X;\mathcal{F}) \to (Y;\mathcal{G})$$

induces a morphism

$$\mathrm{R}\Gamma(Y;\mathcal{G}) \to \mathrm{R}\Gamma(X;\mathcal{F})$$

and, in particular, a canonical morphism

$$(f;\varphi)^* : \mathrm{H}^{\cdot}(Y;\mathcal{G}) \to \mathrm{H}^{\cdot}(X;\mathcal{F}).$$

Moreover, these morphisms are compatible with the composition of morphisms of sheafed spaces.

Proof. We have a functorial morphism

$$\Gamma(Y;\mathcal{G}) \to \Gamma(X;f^{-1}\mathcal{G}) \tag{*}$$

Since the functor f^{-1} is exact, a standard result of homological algebra gives us a morphism

$$\mathrm{R}\Gamma(Y;\mathcal{G}) \to \mathrm{R}\Gamma(X;f^{-1}\mathcal{G}).$$

Composing with the canonical morphism

$$\mathrm{R}\Gamma(X;f^{-1}\mathcal{G}) \to \mathrm{R}\Gamma(X;\mathcal{F})$$

induced by φ, we get the requested morphism. It is possible to visualize this abstract construction more explicitly as follows. Assume we are given a quasi-isomorphism

$$\mathcal{G} \xrightarrow{\alpha} \mathcal{J}^{\cdot}$$

where $\mathcal{J}^{\cdot}$ is a complex of $\Gamma(Y;\cdot)$-acyclic sheaves and a commutative diagram of the form

$$\begin{tikzcd}
f^{-1}(\mathcal{G}) \arrow[r, "f^{-1}(\alpha)"] \arrow[d, "\varphi"'] & f^{-1}(\mathcal{J}^{\cdot}) \arrow[d, "\psi"] \\
\mathcal{F} \arrow[r, "\beta"'] & \mathcal{I}^{\cdot}
\end{tikzcd}$$

where β is a quasi-isomorphism and $\mathcal{I}^{\cdot}$ is a complex of $\Gamma(X;\cdot)$-acyclic sheaves. Then, in $\mathcal{D}^+(\mathcal{A}b)$, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma(Y;\mathcal{G}) \arrow[r, "(1)", "\sim"'] \arrow[dd, "(2)"'] & \Gamma(Y;\mathcal{J}^{\cdot}) \arrow[d, "(3)"] \\
& \Gamma(X;f^{-1}(\mathcal{J}^{\cdot})) \arrow[d, "(4)"] \\
\mathrm{R}\Gamma(X;\mathcal{F}) \arrow[r, "\sim", "(5)"'] & \Gamma(X;\mathcal{I}^{\cdot})
\end{tikzcd}$$

where (1) and (5) are induced by α and β, (3) is induced by morphism of the type (*), (4) is induced by ψ and (2) is the morphism defined abstractly above. □

Examples 1.7.8.

(a) Let $f : X \to Y$ be a morphism of topological spaces. By applying the preceding proposition to the canonical morphism of sheafed spaces

$$(f; f^*) : (X; M_X) \to (Y; M_Y)$$

we get a canonical morphism

$$\mathrm{R}\Gamma(Y; M_Y) \to \mathrm{R}\Gamma(X; M_X)$$

in $\mathcal{D}^+(\mathcal{A}b)$. The associated morphism

$$(f; f^*)^* : \mathrm{H}^\cdot(Y; M_Y) \to \mathrm{H}^\cdot(X; M_X)$$

is often simply denoted by f^*. Clearly, $f^* = \mathrm{id}$ if f is the identity of X. Moreover, if $g : Y \to Z$ is another morphism of topological spaces, we have $f^* \circ g^* = (g \circ f)^*$. This shows that $X \mapsto \mathrm{H}^\cdot(X; M_X)$ is a functor on the category of topological spaces. In particular, $X \simeq Y$ implies $\mathrm{H}^\cdot(X; M_X) \simeq \mathrm{H}^\cdot(Y; M_Y)$.

(b) Let $f : X \to Y$ be a morphism of differential manifolds. It is well known that the pull-back of differential forms is compatible with the exterior differential. In other words, we have

$$d(f^*(\omega)) = f^*(d\omega)$$

for any $\omega \in \Gamma(Y; \mathcal{C}^p_{\infty,Y})$. Using what has been recalled in the proof of the preceding proposition, we see that we have the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^p(Y; \mathbb{C}_Y) & \xrightarrow{f^*} & \mathrm{H}^p(X; \mathbb{C}_X) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{H}^p(\Gamma(Y; \mathcal{C}^\cdot_{\infty,Y})) & \xrightarrow[f^*]{} & \mathrm{H}^p(\Gamma(X; \mathcal{C}^\cdot_{\infty,X})) \end{array}$$

where the first horizontal arrow is the one defined in (a) and the two vertical isomorphisms come from the de Rham theorem. Thanks to what has been said in (a), we see also that, up to isomorphism, de Rham cohomology of X

$$\mathrm{H}^\cdot(\Gamma(X; \mathcal{C}^\cdot_{\infty,X}))$$

depends only on the topology of X and not on its differential structure.

Definition 1.7.9. Let $f : X \to Y$ be a morphism of topological spaces and let $\mathcal{F}$ be an abelian sheaf on X.

The *direct image* of the abelian sheaf $\mathcal{F}$ by f is the sheaf $f(\mathcal{F})$ on Y defined by setting

$$f(\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$$

for any open subset V of Y; the restriction morphisms being the obvious ones.

Example 1.7.10. Let $a_X : X \to \{\mathrm{pt}\}$ be the morphism which maps the topological space X to a point. Identifying sheaves on $\{\mathrm{pt}\}$ with their abelian group of global sections, we have

$$a_X(\mathcal{F}) \simeq \Gamma(X; \mathcal{F})$$

for any abelian sheaf $\mathcal{F}$ on X.

Proposition 1.7.11 (Adjunction formula). *We have a canonical functorial isomorphism*

$$\mathrm{Hom}_{\mathcal{S}hv(Y)}(\mathcal{G}, f(\mathcal{F})) \simeq \mathrm{Hom}_{\mathcal{S}hv(X)}(f^{-1}(\mathcal{G}), \mathcal{F}).$$

This isomorphism is induced by two canonical functorial morphisms

$$f^{-1}(f(\mathcal{F})) \to \mathcal{F}$$

and

$$\mathcal{G} \to f(f^{-1}(\mathcal{G})).$$

Proof. We will only recall the construction of the canonical functorial morphism

$$\mathrm{Hom}_{\mathcal{S}hv(Y)}(\mathcal{G}, f(\mathcal{F})) \to \mathrm{Hom}_{\mathcal{S}hv(X)}(f^{-1}(\mathcal{G}), \mathcal{F}) \qquad (*)$$

Let $h : \mathcal{G} \to f(\mathcal{F})$ be a morphism of abelian sheaves. For any open subset V of Y, we get a morphism

$$h(V) : \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V))$$

and using Remark 1.1.6, it is easy to deduce from these morphisms a morphism

$$h_{x,f(x)} : \mathcal{G}_{f(x)} \to \mathcal{F}_x$$

for any $x \in X$. Now, let U be an open subset of X and let $\sigma \in f^{-1}(\mathcal{G})(U)$. One checks directly that $(h_{x,f(x)}(\sigma_x))_{x\in U} \in \mathcal{F}(U)$. Hence, for any open subset U of X, we have a morphism

$$h'(U) : f^{-1}(\mathcal{G})(U) \to \mathcal{F}(U).$$

These morphisms give rise to the morphism

$$h' : f^{-1}(\mathcal{G}) \to \mathcal{F}$$

image of h by (*). □

Proposition 1.7.12. *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of topological spaces. Then,*

$$g(f(\mathcal{F})) \simeq (g \circ f)(\mathcal{F})$$

canonically and functorially for $\mathcal{F} \in \mathcal{S}hv(X)$. Moreover,

$$(\mathrm{id}_X)(\mathcal{F}) \simeq \mathcal{F}$$

canonically and functorially for $\mathcal{F} \in \mathcal{S}hv(X)$.

Proposition 1.7.13. *The direct image functor*

$$f : \mathcal{S}hv(X) \to \mathcal{S}hv(Y)$$

is left exact and has a right derived functor

$$Rf : \mathcal{D}^+(\mathcal{S}hv(X)) \to \mathcal{D}^+(\mathcal{S}hv(Y)).$$

Moreover, if $g : Y \to Z$ is another morphism of topological spaces, then

$$Rg \circ Rf \simeq R(g \circ f).$$

Example 1.7.14. Denoting $a_X : X \to \{\mathrm{pt}\}$ the canonical map, we deduce from the functorial isomorphism

$$a_X(\mathcal{F}) \simeq \Gamma(X; \mathcal{F})$$

that

$$Ra_X(\cdot) \simeq \mathrm{R}\Gamma(X; \cdot).$$

Therefore, the second part of the preceding proposition contains the fact that

$$\mathrm{R}\Gamma(Y; Rf(\mathcal{F})) \simeq \mathrm{R}\Gamma(X; \mathcal{F});$$

a result which replaces Leray's spectral sequence in the framework of derived categories.

Proposition 1.7.15. *Assume that*

(a) the map f is closed (i.e. such that $f(F)$ is a closed subset of Y if F is a closed subset of X);

(b) the fiber $f^{-1}(y)$ is a taut subspace of X for any $y \in f(Y)$.

Then,

$$[Rf(\mathcal{F})]_y = \mathrm{R}\Gamma(f^{-1}(y); \mathcal{F})$$

for any $y \in Y$ and any abelian sheaf $\mathcal{F}$ on X.

Example 1.7.16. Thanks to Proposition 1.5.3, it is clear that the conditions on f in the preceding proposition are satisfied if one of the following conditions holds :

(a) f is closed and X is metrizable;

(b) f is closed, Y is Haussdorf and X is paracompact;

(c) f is proper and X is Haussdorf.

Remark 1.7.17. Note that under the assumptions of Proposition 1.7.15, we have of course

$$[f(\mathcal{F})]_y = \Gamma(f^{-1}(y); \mathcal{F})$$

but that this formula may be false in general.

Corollary 1.7.18 (Vietoris-Begle). *Assume that*

(a) the map f is closed and surjective,

(b) the fiber $f^{-1}(y)$ is a taut subspace of X for any $y \in Y$,

(c) the fiber $f^{-1}(y)$ is connected and acyclic (i.e.

$$\mathrm{H}^k(f^{-1}(y); M_{f^{-1}(y)}) \simeq 0$$

for any $k > 0$ and any abelian group M) for any $y \in Y$.

Then, the canonical morphism

$$\mathcal{G} \to Rf(f^{-1}(\mathcal{G}))$$

is an isomorphism for any $\mathcal{G} \in \mathcal{D}^+(\mathcal{S}hv(Y))$. In particular, the canonical morphism

$$\mathrm{R}\Gamma(Y; \mathcal{G}) \to \mathrm{R}\Gamma(X; f^{-1}\mathcal{G})$$

is an isomorphism in $\mathcal{D}^+(\mathcal{A}b)$.

Proof. It is sufficient to note that

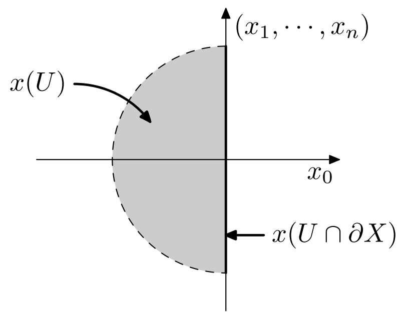
$$(f^{-1}\mathcal{G})_{|f^{-1}(y)} \simeq (\mathcal{G}_y)_{f^{-1}(y)}$$

and that the canonical morphism

$$M \to \mathrm{H}^0(f^{-1}(y); M_{f^{-1}(y)})$$

is an isomorphism if $f^{-1}(y)$ is non-empty and connected. □

Figure 1.7.1:



Remark 1.7.19. Let us recall a few facts about Stokes' theorem which are needed in the following exercise. Let X be an oriented n-dimensional differential manifold with boundary. As is well-known, the orientation of X induces an orientation on ∂X. This orientation is characterized by the fact that if $x : U \to \mathbb{R}^n$ is a positively oriented local coordinate system of X on an open neighborhood of $u \in \partial X$ such that

$$x(u) = 0, \quad x(U) = \{x \in B_n : x_0 \leq 0\}, \quad x(U \cap \partial X) = \{x \in B_n : x_0 = 0\};$$

(see Figure 1.7.1) then $(x_1, \cdots, x_n)_{|U \cap \partial X}$ is a positively oriented coordinate system of ∂X. With this orientation of ∂X, Stokes' formula states that

$$\int_X d\omega = \int_{\partial X} \omega$$

for any $\omega \in \Gamma_c(X; \mathcal{C}^{n-1}_{\infty,X})$.

Exercise 1.7.20.

(a) *Let X be an oriented n-dimensional differential manifold. Show that integration of smooth compactly supported n-forms induces a morphism*

$$\int_X : \mathrm{H}^n_c(X; \mathbb{C}) \to \mathbb{C}$$

(b) *Let X be an oriented n-dimensional differential manifold with boundary. Express the various morphisms of the excision long exact sequence*

$$\begin{tikzcd}
\cdots \mathrm{H}^k_c(X \setminus \partial X; \mathbb{C}) \arrow[r, "u^k"] & \mathrm{H}^k_c(X; \mathbb{C}) \arrow[r, "v^k"] & \mathrm{H}^k_c(\partial X; \mathbb{C}) \arrow[dll, "\delta^k"] \\
\mathrm{H}^{k+1}_c(X \setminus \partial X; \mathbb{C}) \arrow[r, "u^{k+1}"] & \mathrm{H}^{k+1}_c(X; \mathbb{C}) \arrow[r, "v^{k+1}"] & \mathrm{H}^{k+1}_c(\partial X; \mathbb{C}) \cdots
\end{tikzcd}$$

in terms of de Rham cohomology. Show in particular that

$$\int_X \delta^{n-1} c^{n-1} = \int_{\partial X} c^{n-1}$$

for any $c^{n-1} \in \mathrm{H}_c^{n-1}(\partial X; \mathbb{C})$.

Solution. (a) Integration gives us a morphism

$$\int_X : \Gamma_c(X; \mathcal{C}^n_{\infty,X}) \to \mathbb{C}.$$

By Stokes' theorem, we know that

$$\int_X d\omega = \int_{\partial X} \omega = 0$$

for any $\omega \in \Gamma_c(X; \mathcal{C}^{n-1}_{\infty,X})$. Therefore, $\int_X$ induces a morphism

$$\int_X : \Gamma_c(X; \mathcal{C}^n_{\infty,X})/d\Gamma_c(X; \mathcal{C}^n_{\infty,X}) \to \mathbb{C}$$

and the conclusion follows from the de Rham theorem.

(b) Since $\mathcal{C}^{\cdot}_{\infty,X}$ is a c-soft resolution of the sheaf $\mathbb{C}_X$, the long exact sequence of cohomology comes from the application of the snake's lemma to the exact sequence of complexes

$$0 \to \Gamma_c(X \setminus \partial X; \mathcal{C}^{\cdot}_{\infty,X\,|X\setminus\partial X}) \to \Gamma_c(X; \mathcal{C}^{\cdot}_{\infty,X}) \to \Gamma_c(\partial X; \mathcal{C}^{\cdot}_{\infty,X\,|\partial X}) \to 0$$

The canonical restriction morphism

$$\mathcal{C}^{\cdot}_{\infty,X\,|X\setminus\partial X} \to \mathcal{C}^{\cdot}_{\infty,X\setminus\partial X}$$

is clearly an isomorphism. As for the restriction morphism

$$\mathcal{C}^{\cdot}_{\infty,X\,|\partial X} \to \mathcal{C}^{\cdot}_{\infty,\partial X}$$

it is a quasi-isomorphism since in the commutative diagram of complexes

$$\begin{array}{ccc} \mathbb{C}_{X\,|\partial X} & \xrightarrow{\sim} & \mathbb{C}_{\partial X} \\ \downarrow & & \downarrow \\ \mathcal{C}^{\cdot}_{\infty,X\,|\partial X} & \to & \mathcal{C}^{\cdot}_{\infty,X} \end{array}$$

both vertical arrows are quasi-isomorphisms. It follows that

$$\Gamma_c(X \setminus \partial X; \mathcal{C}^{\cdot}_{\infty,X\,|X\setminus\partial X}) \simeq \Gamma_c(X \setminus \partial X; \mathcal{C}^{\cdot}_{\infty,X\setminus\partial X})$$

and that

$$\Gamma_c(\partial X; \mathcal{C}^{\cdot}_{\infty,X\,|\partial X}) \underset{\text{qis}}{\simeq} \Gamma(\partial X; \mathcal{C}^{\cdot}_{\infty,\partial X}).$$

Computation of u^k. Let $c^k \in \mathrm{H}^k_c(X \setminus \partial X; \mathbb{C})$ be represented by

$$\omega^k \in \Gamma_c(X \setminus \partial X; \mathcal{C}^k_{\infty,X\setminus\partial X}).$$

By extension by zero, ω^k becomes a class

$$\omega'^k \in \Gamma_c(X; \mathcal{C}^k_{\infty,X})$$

and $u^k(c^k)$ is represented by ω'^k.

Computation of v^k. Let $c^k \in \mathrm{H}^k_c(X; \mathbb{C})$ be represented by

$$\omega^k \in \Gamma_c(X; \mathcal{C}^k_{\infty,X})$$

and let $j : \partial X \to X$ denotes the inclusion map. Then, $v^k(\omega^k)$ is represented by

$$j^*(\omega^k) \in \Gamma_c(\partial X; \mathcal{C}^k_{\infty,\partial X}).$$

Computation of δ^k. Let $c^k \in \mathrm{H}^k(\partial X; \mathbb{C})$ be represented by

$$\omega^k \in \Gamma_c(\partial X; \mathcal{C}^k_{\infty,\partial X}).$$

Set $K = \operatorname{supp} \omega^k$. We know from elementary differential geometry (collar neighborhood theorem) that there is a neighborhood U of ∂X in X, a differentiable map $p : U \to \partial X$ such that $p \circ j = \mathrm{id}_{\partial X}$ and a smooth function φ equal to 1 on a neighborhood of ∂X and such that $\operatorname{supp} \varphi$ is a p-proper subset U. Denote ω'^k the k-form on X obtained by extending $\varphi p^* \omega$ by zero outside of U. Clearly, ω'^k has compact support and

$$j^* \omega'^k = \omega^k.$$

Therefore, it follows from the snake's lemma that $\delta^k(c^k)$ is represented by $d\omega'^k_{|X\setminus\partial X}$. As expected, this form has compact support. As a matter of fact,

$$d\omega'^k_{|U} = d\varphi \wedge p^*\omega^k + \varphi p^* d\omega^k = d\varphi \wedge p^*\omega^k$$

and $d\varphi = 0$ in a neighborhood of ∂X. Assuming now that $k = n - 1$, we see using Stokes' theorem that

$$\begin{aligned} \int_{X\setminus\partial X} \delta^{n-1} c^{n-1} &= \int_X d\omega'^{n-1} = \int_{\partial X} \omega'^{n-1} = \int_{\partial X} j^* p^* \omega^{n-1} \\ &= \int_{\partial X} \omega^{n-1} = \int_{\partial X} c^{n-1}. \end{aligned}$$

□

1.8 Homotopy theorem

Definition 1.8.1. Let $\mathcal{F}$, $\mathcal{G}$ be abelian sheaves on X and Y respectively and let

$$(h;\psi) : (X \times I; p_X^{-1}\mathcal{F}) \to (Y;\mathcal{G})$$

be a morphism where $I = [0,1]$ and $p_X : X \times I \to X$ denotes the first projection. Let $t \in [0,1]$. We denote

$$i_t : X \to X \times I$$

the morphism defined by setting $i_t(x) = (x,t)$ and h_t the morphism $h \circ i_t$. Applying i_t^{-1} to

$$\psi : h^{-1}\mathcal{G} \to p_X^{-1}\mathcal{F}$$

we get a morphism

$$i_t^{-1}\psi : i_t^{-1}h^{-1}\mathcal{G} \to i_t^{-1}p_X^{-1}\mathcal{F}.$$

Since $h \circ i_t = h_t$ and $p_X \circ i_t = \mathrm{id}_X$, this gives us a morphism

$$\psi_t : h_t^{-1}\mathcal{G} \to \mathcal{F}$$

and a corresponding morphism

$$(h_t;\psi_t) : (X;\mathcal{F}) \to (Y;\mathcal{G}).$$

We call $(h;\psi)$ a *homotopy* between

$$(h_0;\psi_0) : (X;\mathcal{F}) \to (Y,\mathcal{G})$$

and

$$(h_1;\psi_1) : (X;\mathcal{F}) \to (Y;\mathcal{G}).$$

Two morphisms of abelian sheafed spaces connected by a homotopy are said to be *homotopic.*

Proposition 1.8.2. *Let $\mathcal{F}$, $\mathcal{G}$ be abelian sheaves on X and Y. Assume the morphisms*

$$(f_0;\varphi_0) : (X;\mathcal{F}) \to (Y;\mathcal{G})$$

and

$$(f_1;\varphi_1) : (X;\mathcal{F}) \to (Y;\mathcal{G})$$

are homotopic. Then, the morphisms

$$(f_0;\varphi_0)^* : \mathrm{H}^{\cdot}(Y;\mathcal{G}) \to \mathrm{H}^{\cdot}(X;\mathcal{F})$$

and

$$(f_1;\varphi_1)^* : \mathrm{H}^{\cdot}(Y;\mathcal{G}) \to \mathrm{H}^{\cdot}(X;\mathcal{F})$$

are equal.

Proof. Since the application

$$p_X : X \times I \to X$$

is proper and surjective and I is connected and acyclic (see Exercise 1.6.7), Corollary 1.7.18 shows that the canonical morphism

$$(p_X; \pi_X) : (X \times I; p_X^{-1}\mathcal{F}) \to (X; \mathcal{F})$$

induces the isomorphism

$$(p_X; \pi_X)^* : \mathrm{H}^{\cdot}(X; \mathcal{F}) \xrightarrow{\sim} \mathrm{H}^{\cdot}(X \times I; p_X^{-1}\mathcal{F}).$$

If

$$(i_t; \iota_t) : (X; \mathcal{F}) \to (X \times I; p_X^{-1}\mathcal{F})$$

denotes the canonical morphism, we have

$$(p_X; \pi_X) \circ (i_t; \iota_t) = \mathrm{id}\,.$$

It follows that $(i_t; \iota_t)^*$ is the inverse of the isomorphism $(p_X; \pi_X)^*$ and thus does not depend on $t \in [0, 1]$.

Let

$$(h; \psi) : (X \times I; p_X^{-1}\mathcal{F}) \to (Y; \mathcal{G})$$

be a homotopy between $(f_0; \varphi_0)$ and $(f_1; \varphi_1)$. Since

$$(h; \psi) \circ (i_0; \iota_0) = (f_0; \varphi_0), \qquad (h; \psi) \circ (i_1; \iota_1) = (f_1; \varphi_1),$$

we see that

$$(f_0; \varphi_0)^* = (i_0; \iota_0)^* \circ (h; \psi)^* = (i_1; \iota_1)^* \circ (h; \psi)^* = (f_1; \varphi_1)^*.$$

□

Corollary 1.8.3. *If two morphisms*

$$f_0 : X \to Y \qquad \text{and} \qquad f_1 : X \to Y$$

are homotopic in the topological sense, then, for any abelian group M,

$$f_0^* : \mathrm{H}^{\cdot}(Y; M_Y) \to \mathrm{H}^{\cdot}(X; M_X)$$

and

$$f_1^* : \mathrm{H}^{\cdot}(Y; M_Y) \to \mathrm{H}^{\cdot}(X; M_X)$$

are equal. In particular, if X and Y are homotopically equivalent, then

$$\mathrm{H}^{\cdot}(X; M_X) \simeq \mathrm{H}^{\cdot}(Y; M_Y)$$

for any abelian group M.

Examples 1.8.4.

(a) The preceding corollary allows us to show that

$$\mathrm{H}^k(B_n; M_{B_n}) = \begin{cases} M & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

for any abelian group M. As a matter of fact,

$$\begin{aligned} h : B_n \times I &\to B_n \\ (x, t) &\mapsto tx \end{aligned}$$

is a homotopy between the constant map

$$\begin{aligned} h_0 : B_n &\to B_n \\ x &\mapsto 0 \end{aligned}$$

and the identity map

$$\begin{aligned} h_1 : B_n &\to B_n \\ x &\mapsto x \end{aligned}$$

Therefore, the inclusion map $j_0 : \{0\} \to B_n$ and the projection map $p_0 : B_n \to \{0\}$ are inverse of each other in the category of topological spaces modulo homotopy. Hence,

$$\mathrm{H}^{\cdot}(B_n; M_{B_n}) \simeq \mathrm{H}^{\cdot}(\{0\}; M_{\{0\}})$$

and the conclusion follows.

Note that in contrast with Exercise 1.4.10, we have not made use of de Rham theorem. A similar reasoning shows that

$$\mathrm{H}^k(\overline{B}_n; \overline{M}_{\overline{B}_n}) \simeq \begin{cases} M & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

for any abelian group M.

(b) Working as in Exercise 1.6.6 (a) and (b), we can deduce from (a) that, for $n \geq 1$ and any abelian group M, we have

$$\mathrm{H}^k(S_n; M_{S_n}) \simeq \begin{cases} M & \text{if } k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{H}^k_c(B_n; M_{B_n}) \simeq \begin{cases} M & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

Exercise 1.8.5. *Assume $n \geq 1$. Endow $\overline{B}_{n+1}$ with the orientation induced by the canonical orientation of $\mathbb{R}^{n+1}$ and orient S_n as $\partial\overline{B}_{n+1}$. Denote $\int$ the morphism obtained by composing the morphism*

$$\mathrm{H}^n(S_n;\mathbb{Z}) \to \mathrm{H}^n(S_n;\mathbb{C})$$

induced by the inclusion of $\mathbb{Z}$ in $\mathbb{C}$ with the integral

$$\int_{S_n} : \mathrm{H}^n(S_n;\mathbb{C}) \to \mathbb{C}$$

of Exercise 1.7.20. Show that the group

$$\mathrm{H}^n(S_n;\mathbb{Z})$$

has a unique generator v_{S_n} such that $\int v_{S_n} = 1$ and that $\int$ induces an isomorphism between $\mathrm{H}^n(S_n;\mathbb{Z})$ and $\mathbb{Z}$.

Solution. Since $\mathrm{H}^n(S_n;\mathbb{Z}) \simeq \mathbb{Z}$, this group has only two generators g_1 and $g_2 = -g_1$. The uniqueness is thus obvious. Let us prove the existence. We shall use the same notations as in Exercise 1.6.6. It is clear that we have the following canonical morphism of distinguished triangles

$$\begin{tikzcd}
\mathrm{R}\Gamma_c(S_n;\mathbb{Z}) \arrow[r] & \mathrm{R}\Gamma_c(S_n^+;\mathbb{Z}) \oplus \mathrm{R}\Gamma_c(S_n^-;\mathbb{Z}) \arrow[r] & \mathrm{R}\Gamma_c(S_{n-1};\mathbb{Z}) \arrow[r, "+1"] & {} \\
\mathrm{R}\Gamma_c(S_n^+ \setminus S_{n-1};\mathbb{Z}) \arrow[r] \arrow[u] & \mathrm{R}\Gamma_c(S_n^+;\mathbb{Z}) \arrow[r] \arrow[u, "\binom{\mathrm{id}}{0}"] & \mathrm{R}\Gamma_c(S_{n-1};\mathbb{Z}) \arrow[r, "+1"] \arrow[u, "\mathrm{id}"'] & {}
\end{tikzcd}$$

where the first vertical arrow is induced by zero extension of sections. It follows that we have the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^{n-1}(S_{n-1};\mathbb{Z}) \arrow[r, "\delta'^{n-1}"] \arrow[d, equal] & \mathrm{H}^n(S_n;\mathbb{Z}) \\
\mathrm{H}^{n-1}(S_{n-1};\mathbb{Z}) \arrow[r, "\delta^{n-1}"'] & \mathrm{H}^n_c(S_n^+ \setminus S_{n-1};\mathbb{Z}) \arrow[u]
\end{tikzcd}$$

where δ'^k and δ^k are the Mayer-Vietoris and excision "coboundary operators". Since S_n^+ is an oriented manifold with boundary, we get from Exercise 1.7.20 that

$$\int_{S_n} \delta'^{n-1}(c^{n-1}) = \int_{S_n^+ \setminus S_{n-1}} \delta^{n-1}(c^{n-1}) = \pm \int_{S^{n-1}} c^{n-1}$$

where the sign $+$ appears if S^{n-1} is oriented as the boundary of S_n^+ and the sign $-$ appears in the other case (a simple computation shows that the sign is in fact $(-1)^n$).

For $n = 1$, we have the commutative diagram

$$\begin{tikzcd} \mathrm{H}^0(S_1^+;\mathbb{Z}) \oplus \mathrm{H}^0(S_1^-;\mathbb{Z}) \arrow[r] \arrow[d, "\wr"] & \mathrm{H}^0(S_0;\mathbb{Z}) \arrow[r, "\delta'^0"] \arrow[d, "\wr"] & \mathrm{H}^1(S_1;\mathbb{Z}) \arrow[d, "\wr"] \\ \mathbb{Z}^2 \arrow[r, "\left(\begin{smallmatrix} 1 & -1 \\ 1 & -1 \end{smallmatrix}\right)"'] & \mathbb{Z}^2 \arrow[r, "(1\ -1)"'] & \mathbb{Z} \end{tikzcd}$$

Therefore, the generators of $\mathrm{H}^1(S_1;\mathbb{Z})$ are the images by δ'^0 of the functions φ_1 and φ_2 defined on S_0 by setting

$$\begin{cases} \varphi_1(-1) = 0 \\ \varphi_1(1) = 1 \end{cases} \qquad \text{and} \qquad \begin{cases} \varphi_2(-1) = 1 \\ \varphi_2(1) = 0 \end{cases}$$

Since

$$\int_{S_1} \delta'^0 \varphi_2 = -(\varphi_2(1) - \varphi_2(-1)) = 1$$

we may take $v_{S_1} = \delta'^0 \varphi_2$.

Assume now that $n > 1$ and that we have found $v_{S_{n-1}} \in \mathrm{H}^{n-1}(S_{n-1};\mathbb{Z})$ such that

$$\int_{S_{n-1}} v_{S_{n-1}} = 1.$$

Since

$$\int_{S_n} \delta'^{n-1} v_{S_{n-1}} = (-1)^n \int_{S_{n-1}} v_{S_{n-1}} = (-1)^n$$

we may choose $v_{S_n} = (-1)^n \delta'^{n-1} v_{S_{n-1}}$. The conclusion follows easily by induction. ☐

Exercise 1.8.6. *Assume $n \geq 1$. Show that if C_n is an open cell of $\mathbb{R}^n$ (i.e. an open subset of $\mathbb{R}^n$ which is homeomorphic to B_n) then*

$$\mathrm{H}^k_c(C_n;\mathbb{Z}) \simeq 0$$

for $k \neq n$ and $\int$ induces an isomorphism

$$\mathrm{H}^n_c(C_n;\mathbb{Z}) \simeq \mathbb{Z}.$$

In particular, there is a unique class $v_{C_n} \in \mathrm{H}^n_c(C_n;\mathbb{Z})$ such that

$$\int v_{C_n} = 1.$$

Show also that if C'_n is an open cell of $\mathbb{R}^n$ included in C_n, then the canonical morphism

$$\mathrm{H}^n_c(C'_n;\mathbb{Z}) \to \mathrm{H}^n_c(C_n;\mathbb{Z})$$

is an isomorphism which sends $v_{C'_n}$ to v_{C_n}.

Solution. Let us first assume that $C_n = B_n$. We start from the excision distinguished triangle

$$\mathrm{R}\Gamma_c(B_n;\mathbb{Z}) \to \mathrm{R}\Gamma(\overline{B}_n;\mathbb{Z}) \to \mathrm{R}\Gamma(S_{n-1};\mathbb{Z}) \xrightarrow{+1}$$

For $n = 1$, we get the exact sequence

$$\begin{tikzcd}
\mathrm{H}^0(\overline{B}_n;\mathbb{Z}) \arrow[r] \arrow[d, "\wr"] & \mathrm{H}^1(S_0;\mathbb{Z}) \arrow[r, "\delta^0"] \arrow[d, "\wr"] & \mathrm{H}^1_c(B_1;\mathbb{Z}) \arrow[r] \arrow[d, "\wr"] & 0 \\
\mathbb{Z} \arrow[r, "\binom{1}{1}"'] & \mathbb{Z}^2 \arrow[r, "(-1\ 1)"'] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

It follows that a generator of $\mathrm{H}^1_c(B_1;\mathbb{Z})$ is $v_{B_1} = \delta^0([\varphi])$ where

$$\begin{cases} \varphi(-1) = 0 \\ \varphi(1) = 1 \end{cases}$$

Since

$$\int_{B_1} v_{B_1} = \int_{B_1} \delta^0([\varphi]) = \varphi(1) - \varphi(-1) = 1,$$

we see that $\int_{B_1}$ induces the isomorphism

$$\int_{B_1} : \mathrm{H}^1_c(B_1;\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}.$$

For $n \geq 2$, we get the isomorphism

$$\begin{tikzcd}
\mathrm{H}^{n-1}(S_{n-1};\mathbb{Z}) \arrow[r, "\delta^{n-1}", "\sim"'] \arrow[d, "\wr"] & \mathrm{H}^n_c(B_n;\mathbb{Z}) \arrow[d, "\wr"] \\
\mathbb{Z} \arrow[r, "1"'] & \mathbb{Z}
\end{tikzcd}$$

Setting $v_{B_n} = \delta^{n-1} v_{S_{n-1}}$, we see that

$$\int_{B_n} v_{B_n} = \int_{B_n} \delta^{n-1} v_{S_{n-1}} = \int_{S_{n-1}} v_{s_{n-1}} = 1$$

and the conclusion follows.

Assume now C_n is a general open cell of $\mathbb{R}^n$. By functoriality, it is clear that

$$\mathrm{H}^k_c(C_n;\mathbb{Z}) \simeq \mathrm{H}^k_c(B_n;\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

Let B be an open ball of $\mathbb{R}^n$ with center x. By inversion, $\mathbb{R}^n \setminus B \simeq \overline{B} \setminus \{x\}$. Therefore,

$$\mathrm{R}\Gamma_c(\mathbb{R}^n \setminus B;\mathbb{Z}) \simeq \mathrm{R}\Gamma_c(\overline{B} \setminus \{x\};\mathbb{Z})$$

From the excision distinguished triangle

$$\mathrm{R}\Gamma_c(\overline{B} \setminus \{x\}; \mathbb{Z}) \to \mathrm{R}\Gamma(\overline{B}; \mathbb{Z}) \to \mathrm{R}\Gamma(\{x\}; \mathbb{Z}) \xrightarrow{+1}$$

it follows that $\mathrm{R}\Gamma_c(\overline{B} \setminus \{x\}; \mathbb{Z}) \simeq 0$. Hence, $\mathrm{R}\Gamma_c(\mathbb{R}^n \setminus B; \mathbb{Z}) \simeq 0$ and

$$\mathrm{R}\Gamma_c(B; \mathbb{Z}) \to \mathrm{R}\Gamma_c(\mathbb{R}^n; \mathbb{Z})$$

is an isomorphism in the derived category.

It follows that

$$\mathrm{H}^n_c(B; \mathbb{Z}) \to \mathrm{H}^n_c(\mathbb{R}^n; \mathbb{Z})$$

is an isomorphism. If we assume moreover that $x \in C_n$ and $B \subset C_n$, we see from the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^n_c(B; \mathbb{Z}) \arrow[rr] \arrow[dr] & & \mathrm{H}^n_c(C_n; \mathbb{Z}) \arrow[dl] \\
& \mathrm{H}^n_c(\mathbb{R}^n; \mathbb{Z}) &
\end{tikzcd}$$

that

$$\mathrm{H}^n_c(C_n; \mathbb{Z}) \to \mathrm{H}^n_c(\mathbb{R}^n; \mathbb{Z})$$

is surjective. Since these groups are both isomorphic to $\mathbb{Z}$, the preceding morphism is in fact an isomorphism. Using the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{H}^n_c(B_n; \mathbb{Z}) \arrow[rr, "\sim"] \arrow[dr, "\int_{B_n}"'] & & \mathrm{H}^n_c(\mathbb{R}^n; \mathbb{Z}) \arrow[dl, "\int_{\mathbb{R}^n}"] \\
& \mathbb{C} &
\end{tikzcd}$$

we see that $\int_{\mathbb{R}^n}$ induces an isomorphism

$$\int_{\mathbb{R}^n} : \mathrm{H}^n_c(\mathbb{R}^n; \mathbb{Z}) \to \mathbb{Z}.$$

A similar argument with B_n replaced by C_n allows us to conclude. $\square$

Exercise 1.8.7. *Let $\varphi : C_n \to C'_n$ be a diffeomorphism between two open cells of $\mathbb{R}^n$. Show that*

$$\varphi^* v_{C'_n} = \mathrm{sgn}(J_\varphi) v_{C_n}$$

where J_φ denotes the Jacobian of φ.

Solution. It follows from the preceding exercise that

$$\varphi^* v_{C'_n} = m \, v_{C_n}.$$

On one hand, we have

$$\int m\, v_{C_n} = m \int v_{C_n} = m.$$

On the other hand, using de Rham cohomology, we get

$$\int \varphi^* v_{C'_n} = \operatorname{sgn}(J_\varphi) \int v_{C'_n} = \operatorname{sgn}(J_\varphi).$$

The conclusion follows. □

Exercise 1.8.8.

(a) *Show that the canonical morphism*

$$\mathrm{R}\Gamma_{\{0\}}(B_n;\mathbb{Z}) \to \mathrm{R}\Gamma_c(B_n;\mathbb{Z})$$

is an isomorphism.

(b) *Deduce from (a) that for any open neighborhood U of u in $\mathbb{R}^n$ we have*

$$\mathrm{H}^k_{\{u\}}(U;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = n; \\ 0 & \text{otherwise.} \end{cases}$$

Show, moreover, that $\mathrm{H}^n_{\{u\}}(U;\mathbb{Z})$ has a unique generator v_u such that

$$\int_U v_u = 1$$

(c) *Let $\varphi : U \to V$ be a diffeomorphism between open subsets of $\mathbb{R}^n$ and denote J_φ its Jacobian. Show that for any $u \in U$, the map*

$$\varphi^* : \mathrm{H}^n_{\{\varphi(u)\}}(V;\mathbb{Z}) \to \mathrm{H}^n_{\{u\}}(U;\mathbb{Z})$$

sends $v_{\varphi(u)}$ to $\operatorname{sgn}(J_\varphi(u))v_u$.

Solution. (a) Consider the morphism of distinguished triangles

$$\begin{array}{ccccccc}
\mathrm{R}\Gamma_{\{0\}}(\overline{B}_n;\mathbb{Z}) & \to & \mathrm{R}\Gamma(\overline{B}_n;\mathbb{Z}) & \to & \mathrm{R}\Gamma(\overline{B}_n \setminus \{0\};\mathbb{Z}) & \xrightarrow{+1} \\
\downarrow & & \downarrow & & \downarrow & \\
\mathrm{R}\Gamma_c(B_n;\mathbb{Z}) & \longrightarrow & \mathrm{R}\Gamma(\overline{B}_n;\mathbb{Z}) & \longrightarrow & \mathrm{R}\Gamma(S_{n-1};\mathbb{Z}) & \xrightarrow{+1}
\end{array}$$

In this diagram, the second vertical arrow is the identity and by the homotopy theorem the third vertical arrow is an isomorphism. It follows that the first vertical arrow is also an isomorphism. Since this arrow is the composition of the isomorphism

$$\mathrm{R}\Gamma_{\{0\}}(\overline{B}_n;\mathbb{Z}) \to \mathrm{R}\Gamma_{\{0\}}(B_n;\mathbb{Z})$$

with the canonical morphism

$$\mathrm{R}\Gamma_{\{0\}}(B_n;\mathbb{Z}) \to \mathrm{R}\Gamma_c(B_n;\mathbb{Z}),$$

we get the conclusion.

(b) The cohomology table follows directly from (a) and Exercise 1.8.6. Let $B_U \subset U$ be an open ball with center u. By (a),

$$\mathrm{H}^n_{\{u\}}(U;\mathbb{Z}) \simeq \mathrm{H}^n_{\{u\}}(B_U;\mathbb{Z}) \simeq \mathrm{H}^n_c(B_U;\mathbb{Z}).$$

Denote v_u the element of $\mathrm{H}^n_{\{u\}}(U;\mathbb{Z})$ corresponding to $v_{B_U} \in \mathrm{H}^n_c(B_U;\mathbb{Z})$. Since the diagram

$$\begin{array}{ccc} \mathrm{H}^n_{\{u\}}(U;\mathbb{Z}) & \longrightarrow & \mathrm{H}^n_c(U;\mathbb{Z}) \\ \downarrow & & \uparrow \\ \mathrm{H}^n_{\{u\}}(B_U;\mathbb{Z}) & \to & \mathrm{H}^n_c(B_U;\mathbb{Z}) \end{array}$$

is commutative, we get

$$\int_U v_u = \int_{B_U} v_{B_U} = 1.$$

(c) Thanks to (b), the result may be obtained by working as in the preceding exercise. □

Exercise 1.8.9. *Let* $B^-_{n+1} = \{(x_0,\cdots,x_n) \in B_{n+1} : x_0 \leq 0\}$ *and identify* B_n *with*

$$\{(0,x_1,\cdots,x_n) : (x_1,\cdots,x_n) \in B_n\}.$$

(see Figure 1.8.1). Denote $v_{B^-_{n+1}\setminus B_n} \in \mathrm{H}^{n+1}_c(B^-_{n+1} \setminus B_n;\mathbb{Z})$ *and* $v_{B_n} \in \mathrm{H}^n_c(B_n;\mathbb{Z})$ *the classes which have an integral equal to* 1. *Show that*

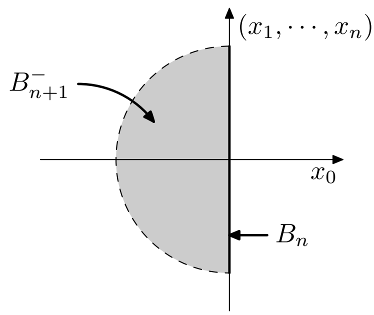
$$v_{B^-_{n+1}\setminus B_n} = \delta^n(v_{B_n})$$

where δ^n *is the coboundary operator associated to the distinguished triangle*

$$\mathrm{R}\Gamma_c(B^-_{n+1} \setminus B_n;\mathbb{Z}) \to \mathrm{R}\Gamma(B^-_{n+1};\mathbb{Z}) \to \mathrm{R}\Gamma_c(B_n;\mathbb{Z}) \xrightarrow{+1}$$

Solution. Thanks to the preceding exercise, this follows directly from Exercise 1.7.20. □

Figure 1.8.1:



1.9 Cohomology of compact polyhedra

Definition 1.9.1. A finite *simplicial complex* is a finite set Σ of non-empty finite sets called *simplexes* such that if $\sigma \in \Sigma$ and $\sigma' \subset \sigma$ then $\sigma' \in \Sigma$. We call the finite set $S = \cup\Sigma$, the *set of vertices* of Σ.

The *dimension of a simplex* $\sigma \in \Sigma$ is the number $\dim \sigma = \#\sigma - 1$. Simplexes of dimension $n \in \mathbb{N}$ are called *n-simplexes.* A *p-face* of an n-simplex σ is a p-simplex σ' such that $\sigma' \subset \sigma$. The *dimension of the finite simplicial complex* Σ is the number

$$\dim \Sigma = \sup_{\sigma \in \Sigma} \dim \sigma.$$

The *n-skeleton* of Σ is the simplicial complex

$$\Sigma_n = \{\sigma \in \Sigma : \dim \sigma \leq n\}.$$

The *realization* of Σ is the compact subspace $|\Sigma|$ of $\mathbb{R}^{\#S}$ defined by setting

$$|\Sigma| = \{\alpha : S \to \mathbb{R} : |\alpha \geq 0, \operatorname{supp} \alpha \in \Sigma, \sum_{s \in S} \alpha(s) = 1\}.$$

If $\sigma \in \Sigma$, we set

$$|\sigma| = \{\alpha \in |\Sigma| \ : \operatorname{supp} \alpha = \sigma\}.$$

Clearly, $|\sigma| \cap |\sigma'| \neq \emptyset$ if and only if $\sigma = \sigma'$ and $|\Sigma| = \bigcup_{\sigma \in \Sigma} |\sigma|$.

The data of a finite simplicial complex Σ and an isomorphism

$$h : |\Sigma| \to X$$

of topological spaces is a finite *triangulation* of X. A topological space X which has a finite triangulation is a *compact polyhedron.*

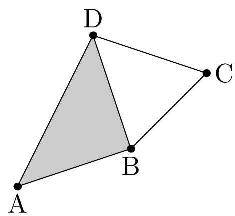
Examples 1.9.2. The set Σ whose elements are

$$\{A\},\{B\},\{C\},\{D\},\{A,B\},\{A,D\},\{B,D\},\{B,C\},\{D,C\},\{A,B,D\}$$

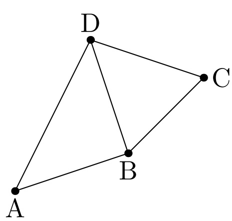
is a finite simplicial complex of dimension 2. It has $\{A,B,C,D\}$ as set of vertices and contains

- four 0-simplexes ($\{A\},\{B\},\{C\},\{D\}$),
- five 1-simplexes ($\{A,B\},\{A,D\},\{B,D\},\{B,C\},\{D,C\}$),
- one 2-simplex ($\{A,B,D\}$).

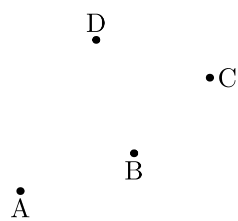
A compact polyhedron homeomorphic to $|\Sigma|$ is



Compact polyhedra homeomorphic to $|\Sigma_1|$ and $|\Sigma_0|$ are respectively



and



Definition 1.9.3. Let σ be a k-simplex of Σ. Two bijections

$$\mu : \{0, \cdots, k\} \to \sigma, \quad \nu : \{0, \cdots, k\} \to \sigma$$

have the same sign if the signature of $\nu^{-1} \circ \mu$ is positive. Clearly, the relation "to have the same sign" is an equivalence relation on the set of bijections between $\{0, \cdots, k\}$ and σ. An equivalence class for this relation is called an *orientation* of σ. Of course, a k-simplex of Σ has only two possible orientations, if o is one of them, we denote $-o$ the other. An *oriented k-simplex* is a k-simplex of Σ endowed with an orientation. If $\mu : \{0, \cdots, k\} \to \sigma$ is a bijection, we denote by $[\mu(0), \cdots, \mu(k)]$ the oriented k-simplex obtained by endowing σ with the orientation associated to μ.

A *k-cochain* of Σ is a map c from the set of oriented k-simplexes of Σ to $\mathbb{Z}$ such that if $c_{(\sigma,o)}$ denotes the values of c on the oriented simplex (σ, o), we have $c_{(\sigma,-o)} = -c_{(\sigma,o)}$. It is clear that k-cochains form a group. We denote it by $C^k(\Sigma)$. We define the *differential*

$$d^k : C^k(\Sigma) \to C^{k+1}(\Sigma)$$

by setting

$$(d^k c)_{[x_0,\cdots,x_{k+1}]} = \sum_{l=0}^{k+1} (-1)^l c_{[x_0,\cdots,\widehat{x}_l,\cdots,x_{k+1}]}.$$

One checks easily that the groups $C^k(\Sigma)$ $(k \geq 0)$ together with the differentials d^k $(k \geq 0)$ form a complex $C^\cdot(\Sigma)$ canonically associated with Σ. We call it the *simplicial cochain complex* of Σ.

Lemma 1.9.4. *Let $\sigma = \{x_0, \cdots, x_k\}$ be a k-simplex of Σ and let α_l be the point of $|\Sigma|$ corresponding to x_l. Set*

$$\varphi_{(x_0,\cdots,x_k)}(t_1, \cdots, t_k) = \alpha_0 + \sum_{l=1}^{k} t_l(\alpha_l - \alpha_0)$$

and

$$J_k = \{(t_1, \cdots, t_k) \in \mathbb{R}^k : t_1 > 0, \cdots, t_k > 0, \sum_{l=1}^{k} t_l < 1\}.$$

Then, J_k is an open cell of $\mathbb{R}^k$ and $\varphi_{(x_0,\cdots,x_k)} : J_k \to |\sigma|$ is an homeomorphism. Moreover, if $v_{(x_0,\cdots,x_k)}$ is the image of v_{J_k} by the isomorphism $\varphi^{-1*}_{(x_0,\cdots,x_k)} : \mathrm{H}^k_c(J_k; \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^k_c(|\sigma|; \mathbb{Z})$, then we have

$$v_{(x_{\mu_0},\cdots,x_{\mu_k})} = (\operatorname{sign} \mu) v_{(x_0,\cdots,x_k)}$$

for any permutation μ of $\{0, \cdots, k\}$.

Proof. The fact that φ is an homeomorphism is obvious. Let μ be a permutation of $\{0, \cdots, k\}$. Set

$$\psi_\mu = \varphi^{-1}_{(x_{\mu_0},\cdots,x_{\mu_k})} \circ \varphi_{(x_0,\cdots,x_k)}.$$

Clearly, ψ_μ is the restriction to J_k of the affinity of $\mathbb{R}^k$ characterized by

$$\psi_\mu(P_l) = P_{\mu_l}$$

where $P_0 = 0$, $P_1 = e_1$, $\cdots$, $P_k = e_k$. It follows that ψ_μ preserves or reverses the orientation of $\mathbb{R}^k$ according to the fact that μ is even or odd. Using Exercise 1.8.7, we see that

$$\psi^*_\mu(v_{J_k}) = (\operatorname{sign} \mu) v_{J_k}.$$

Hence,

$$\begin{aligned}\varphi^{-1*}_{(x_{\mu_0},\cdots,x_{\mu_k})}(v_{J_k}) &= \varphi^{-1*}_{(x_0,\cdots,x_k)} \circ \psi^*_\mu(v_{J_k}) \\ &= (\operatorname{sign}\mu)\varphi^{-1*}_{(x_0,\cdots,x_k)}(v_{J_k})\end{aligned}$$

and the conclusion follows. ☐

Lemma 1.9.5. *Let $\sigma = \{x_0, \cdots, x_k\}$ be a k-simplex of Σ and let $\sigma' = \{x_1, \cdots, x_k\}$. The distinguished triangle*

$$\mathrm{R}\Gamma_c(|\sigma|;\mathbb{Z}) \to \mathrm{R}\Gamma_c(|\sigma| \cup |\sigma'|;\mathbb{Z}) \to \mathrm{R}\Gamma_c(|\sigma'|;\mathbb{Z}) \xrightarrow{+1}$$

induces a canonical morphism

$$\delta^{k-1} : \mathrm{H}^{k-1}_c(|\sigma'|;\mathbb{Z}) \to \mathrm{H}^k_c(|\sigma|;\mathbb{Z})$$

and we have

$$\delta^{k-1}(v_{[x_1,\cdots,x_k]}) = v_{[x_0,\cdots,x_k]}.$$

Proof. The existence of the distinguished triangle follows from the fact that $|\sigma|$ (resp. $|\sigma'|$) is open (resp. closed) in $|\sigma| \cup |\sigma'|$. Thanks to the morphism $\varphi_{(x_0,\cdots,x_k)}$ of the preceding lemma, we may assume that $|\sigma| = J_k$, $|\sigma'| = \{(t_1, \cdots, t_k) : t_1 > 0, \cdots, t_k > 0, \sum_{l=1}^k t_k = 1\}$. Then, $|\sigma| \cup |\sigma'|$ appears as a manifold with boundary and the result follows from Exercises 1.8.9 and 1.7.20 since

$$\begin{vmatrix} 1 & -1 & -1 & \cdots & -1 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{vmatrix} > 0.$$

☐

Proposition 1.9.6. *For any $k \geq 0$, there is a canonical isomorphism between $\mathrm{H}^k(|\Sigma|;\mathbb{Z})$ and $\mathrm{H}^k(C^\cdot(\Sigma))$.*

Proof. Let us consider the excision distinguished triangle

$$\mathrm{R}\Gamma_c(|\Sigma_p| \setminus |\Sigma_{p-1}|;\mathbb{Z}) \to \mathrm{R}\Gamma_c(|\Sigma_p|;\mathbb{Z}) \to \mathrm{R}\Gamma_c(|\Sigma_{p-1}|;\mathbb{Z}) \xrightarrow{+1} \qquad (*)$$

Since $|\Sigma_p| \setminus |\Sigma_{p-1}| = \bigsqcup_{\sigma \in \Sigma_p \setminus \Sigma_{p-1}} |\sigma|$ and $|\sigma|$ is open in $|\Sigma_p|$ for any $\sigma \in \Sigma_p \setminus \Sigma_{p-1}$, we get

$$\mathrm{R}\Gamma_c(|\Sigma_p| \setminus |\Sigma_{p-1}|;\mathbb{Z}) \simeq \bigoplus_{\sigma \in \Sigma_p \setminus \Sigma_{p-1}} \mathrm{R}\Gamma_c(|\sigma|;\mathbb{Z}).$$

Using the fact that $|\sigma|$ is homeomorphic to an open ball of $\mathbb{R}^p$ if $\sigma \in \Sigma_p \setminus \Sigma_{p-1}$, we see that

$$\mathrm{H}^k_c(|\Sigma_p| \setminus |\Sigma_{p-1}|; \mathbb{Z}) = \begin{cases} \displaystyle\bigoplus_{\sigma \in \Sigma_p \setminus \Sigma_{p-1}} \mathrm{H}^p_c(|\sigma|; \mathbb{Z}) & \text{if } k = p \\ 0 & \text{otherwise.} \end{cases}$$

Using Lemma 1.9.4, we get a canonical isomorphism

$$\mathrm{H}^p_c(|\Sigma_p| \setminus |\Sigma_{p-1}|; \mathbb{Z}) \simeq C^p(\Sigma).$$

The long exact sequence of cohomology associated to (*) is

$$\begin{tikzcd} \cdots \mathrm{H}^k_c(|\Sigma_p| \setminus |\Sigma_{p-1}|; \mathbb{Z}) \arrow[r] & \mathrm{H}^k(|\Sigma_p|; \mathbb{Z}) \arrow[r] & \mathrm{H}^k(|\Sigma_{p-1}|; \mathbb{Z}) \arrow[dll] \\ \mathrm{H}^{k+1}_c(|\Sigma_p| \setminus |\Sigma_{p-1}|; \mathbb{Z}) \arrow[r] & \mathrm{H}^{k+1}(|\Sigma_p|; \mathbb{Z}) \arrow[r] & \mathrm{H}^{k+1}(|\Sigma_{p-1}|; \mathbb{Z}) \cdots \end{tikzcd}$$

For $k > p$, we get that

$$\mathrm{H}^k(|\Sigma_p|; \mathbb{Z}) \simeq \mathrm{H}^k(|\Sigma_{p-1}|; \mathbb{Z}).$$

By decreasing induction on p, we see that $\mathrm{H}^k(|\Sigma_p|; \mathbb{Z}) \simeq \mathrm{H}^k(|\Sigma_0|; \mathbb{Z}) \simeq 0$. For $k < p - 1$, we obtain

$$\mathrm{H}^k(|\Sigma_p|; \mathbb{Z}) \simeq \mathrm{H}^k(|\Sigma_{p-1}|; \mathbb{Z}).$$

By increasing induction on p, this gives us the isomorphism

$$\mathrm{H}^k(|\Sigma_{p-1}|; \mathbb{Z}) \simeq \mathrm{H}^k(|\Sigma|; \mathbb{Z}).$$

For $k = p - 1$, we get the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^{p-1}(|\Sigma_p|; \mathbb{Z}) \arrow[r] & \mathrm{H}^{p-1}(|\Sigma_{p-1}|; \mathbb{Z}) \arrow[dll] \\ \mathrm{H}^p_c(|\Sigma_p| \setminus |\Sigma_{p-1}|; \mathbb{Z}) \arrow[r] & \mathrm{H}^p(|\Sigma_p|; \mathbb{Z}) \arrow[r] & \mathrm{H}^p(|\Sigma_{p-1}|; \mathbb{Z}) \end{tikzcd}$$

Using the isomorphisms obtained above, we may rewrite this sequence as

$$0 \to \mathrm{H}^{p-1}(|\Sigma|; \mathbb{Z}) \xrightarrow{\alpha^{p-1}} \mathrm{H}^{p-1}(|\Sigma_{p-1}|; \mathbb{Z}) \xrightarrow{\beta^{p-1}} C^p(|\Sigma|) \xrightarrow{\gamma^p} \mathrm{H}^p(|\Sigma_p|; \mathbb{Z}) \to 0.$$

Set $\delta^p = \beta^p \circ \gamma^p$. Clearly, $\operatorname{Coker} \delta^{p-1} \simeq \operatorname{Coker} \beta^{p-1} \simeq \mathrm{H}^p(|\Sigma_p|; \mathbb{Z})$ and through this isomorphism, the canonical map

$$(\delta^p)' : \operatorname{Coker} \delta^{p-1} \to C^{p+1}(|\Sigma|)$$

becomes the map

$$\beta^p : \mathrm{H}^p(|\Sigma_p|;\mathbb{Z}) \to C^{p+1}(|\Sigma|).$$

It follows that $\operatorname{Ker}\delta^p/\operatorname{Im}\delta^{p-1}$ is canonically isomorphic to

$$\operatorname{Ker}\beta^p \simeq \mathrm{H}^p(|\Sigma|;\mathbb{Z}).$$

To conclude, it remains to show that $\delta^p = d^p$. Let $[x_0,\cdots,x_{p+1}]$ be an oriented $(p+1)$-simplex of Σ and let $c \in C^p(\Sigma)$ be a simplicial p-cochain. We have to show that

$$[\delta^p(c)]_{[x_0,\cdots,x_{p+1}]} = \sum_{l=0}^{p+1}(-1)^l c_{[x_0,\cdots,\widehat{x}_l,\cdots,x_{p+1}]}.$$

Denote Δ the simplicial complex formed by the finite non-empty subsets of

$$\{x_0,\cdots,x_{p+1}\}.$$

Using the morphisms of distinguished triangles

$$\begin{tikzcd} \mathrm{R}\Gamma_c(|\Delta_k|\setminus|\Delta_{k-1}|;\mathbb{Z}) \arrow[r] & \mathrm{R}\Gamma_c(|\Delta_k|;\mathbb{Z}) \arrow[r] & \mathrm{R}\Gamma_c(|\Delta_{k-1}|;\mathbb{Z}) \arrow[r, "+1"] & {} \\ \mathrm{R}\Gamma_c(|\Sigma_k|\setminus|\Sigma_{k-1}|;\mathbb{Z}) \arrow[r] \arrow[u] & \mathrm{R}\Gamma_c(|\Sigma_k|;\mathbb{Z}) \arrow[r] \arrow[u] & \mathrm{R}\Gamma_c(|\Sigma_{k-1}|;\mathbb{Z}) \arrow[r, "+1"] \arrow[u] & {} \end{tikzcd}$$

One sees that we may assume $\Sigma = \Delta$. Moreover, we may assume that

$$c_{[x_0,\cdots,\widehat{x}_l,\cdots,x_{p+1}]} = \begin{cases} 1 & \text{if } l = 0 \\ 0 & \text{otherwise} \end{cases}$$

Set $\sigma = \{x_0,\cdots,x_{p+1}\}$ and $\sigma' = \{x_1,\cdots,x_{p+1}\}$. Since $|\sigma|\cup|\sigma'|$ is open in $|\Delta|$ and $|\sigma'|$ is open in $|\Delta_{p-1}|$, we have the morphism of distinguished triangles

$$\begin{tikzcd} \mathrm{R}\Gamma_c(|\sigma|;\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{R}\Gamma_c(|\sigma|\cup|\sigma'|;\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{R}\Gamma_c(|\sigma'|;\mathbb{Z}) \arrow[r, "+1"] \arrow[d] & {} \\ \mathrm{R}\Gamma_c(|\Delta|\setminus|\Delta_p|;\mathbb{Z}) \arrow[r] & \mathrm{R}\Gamma_c(|\Delta|;\mathbb{Z}) \arrow[r] & \mathrm{R}\Gamma_c(|\Delta_p|;\mathbb{Z}) \arrow[r, "+1"] & {} \end{tikzcd}$$

Therefore, the diagram

$$\begin{tikzcd} \mathrm{H}^p_c(|\sigma'|;\mathbb{Z}) \arrow[r, "\varphi"] \arrow[d] & \mathrm{H}^{p+1}_c(|\sigma|;\mathbb{Z}) \arrow[dd] \\ \mathrm{H}^p_c(|\Delta_p|\setminus|\Delta_{p-1}|;\mathbb{Z}) \arrow[d] & \\ \mathrm{H}^p(|\Delta_p|;\mathbb{Z}) \arrow[r] & \mathrm{H}^{p+1}_c(|\Delta|\setminus|\Delta_p|;\mathbb{Z}) \end{tikzcd}$$

is commutative. Since c is the image of the orientation class $v_{[x_1,\cdots,x_p]} \in \mathrm{H}^p_c(|\sigma'|;\mathbb{Z})$ by the canonical morphism

$$\mathrm{H}^p_c(|\sigma'|;\mathbb{Z}) \to \mathrm{H}^p_c(|\Delta_p| \setminus |\Delta_{p-1}|;\mathbb{Z}) \simeq C^p(\Sigma),$$

$\delta^p(c)$ is the image of $\varphi(v_{[x_1,\cdots,x_p]})$ by the canonical morphism

$$\mathrm{H}^{p+1}_c(|\sigma|;\mathbb{Z}) \to \mathrm{H}^{p+1}_c(|\Delta| \setminus |\Delta_p|;\mathbb{Z}) \simeq C^{p+1}(\Sigma).$$

But Lemma 1.9.5 shows that

$$\varphi(v_{[x_1,\cdots,x_p]}) \simeq v_{[x_0,\cdots,x_p]}$$

and the conclusion follows. □

Corollary 1.9.7 (Euler's theorem). *Let Σ be a finite simplicial complex. Then, the abelian groups*

$$\mathrm{H}^k(|\Sigma|;\mathbb{Z}) \quad k \geq 0$$

are finitely generated and the Euler-Poincaré characteristic

$$\chi(|\Sigma|) = \sum_k (-1)^k \operatorname{rk} \mathrm{H}^k(|\Sigma|;\mathbb{Z})$$

is equal to

$$\sum_k (-1)^k \#(\Sigma_k \setminus \Sigma_{k-1}).$$

Proof. The first part follows directly from the fact that

$$C^k(\Sigma) \simeq \mathbb{Z}^{\#(\Sigma_k \setminus \Sigma_{k-1})}$$

is a free abelian group with finite rank.

Denote $Z^k(\Sigma)$ and $B^{k+1}(\Sigma)$ the kernel and the image of the differential

$$d^k : C^k(\Sigma) \to C^{k+1}(\Sigma)$$

and set

$$\mathrm{H}^k(\Sigma) = Z^k(\Sigma)/B^k(\Sigma).$$

From the exact sequences

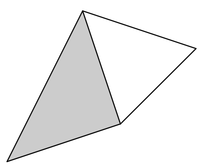
$$0 \to Z^k(\Sigma) \to C^k(\Sigma) \to B^{k+1}(\Sigma) \to 0$$

$$0 \to B^k(\Sigma) \to Z^k(\Sigma) \to \mathrm{H}^k(\Sigma) \to 0$$

we deduce that
$$\operatorname{rk} C^k(\Sigma) = \operatorname{rk} Z^k(\Sigma) + \operatorname{rk} B^{k+1}(\Sigma)$$
and that
$$\operatorname{rk} Z^k(\Sigma) = \operatorname{rk} B^k(\Sigma) + \operatorname{rk} \mathrm{H}^k(\Sigma).$$
It follows that
$$\operatorname{rk} C^k(\Sigma) = \operatorname{rk} B^k(\Sigma) + \operatorname{rk} B^{k+1}(\Sigma) + \operatorname{rk} \mathrm{H}^k(\Sigma)$$
and, hence, that
$$\sum_k (-1)^k \operatorname{rk} C^k(\Sigma) = \sum_k (-1)^k \operatorname{rk} \mathrm{H}^k(\Sigma).$$
The conclusion follows since
$$\operatorname{rk} C^k(\Sigma) = \#(\Sigma_k \setminus \Sigma_{k-1})$$
and
$$\mathrm{H}^k(\Sigma) \simeq \mathrm{H}^k(|\Sigma|; \mathbb{Z}).$$
□

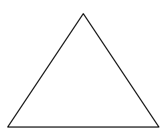
Examples 1.9.8.

(a) Consider a simplicial complex Σ with $|\Sigma|$ homeomorphic to



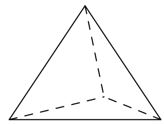
Clearly, Σ has four 0-simplexes, five 1-simplexes and one 2-simplex. Hence,
$$\chi(|\Sigma|; \mathbb{Z}) = 4 - 5 + 1 = 0.$$

(b) Consider a simplicial complex Σ with $|\Sigma|$ homeomorphic to



We have three 0-simplexes and three 1-simplexes. Therefore, $\chi(|\Sigma|; \mathbb{Z}) = 3 - 3 = 0$. Note that $|\Sigma| \simeq S_1$ and our result is compatible with the fact that
$$\mathrm{H}^k(S_1; \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$
since $\dim \mathrm{H}^0(S_1; \mathbb{Z}) - \dim \mathrm{H}^1(S_1; \mathbb{Z}) = 1 - 1 = 0$.

(c) Similarly, for



we have $\chi = 4 - 6 + 4 = 2$. This is compatible with the relation

$$\dim \mathrm{H}^0(S_2; \mathbb{Z}) - \dim \mathrm{H}^1(S_2; \mathbb{Z}) + \dim \mathrm{H}^2(S_2; \mathbb{Z}) = 1 - 0 + 1 = 2.$$

1.10 Cohomology of locally compact spaces

Definition 1.10.1. Let X be a locally compact topological space. We call the cohomological dimension of the functor

$$\mathcal{F} \mapsto \Gamma_c(X; \mathcal{F})$$

the *cohomological dimension* of X and denote it by $\dim_c X$.

Remark 1.10.2. Note that by a well-known result about cohomological dimensions, $\dim_c X \leq n$ $(n \in \mathbb{N})$ if and only if

$$\mathrm{H}^k_c(X; \mathcal{F}) \simeq 0$$

for any $k > n$ and any $\mathcal{F} \in \mathcal{S}hv(X)$. Note also that contrarily to what may appear at first glance, cohomological dimension is a local notion. More precisely, if $\mathcal{U}$ is an open covering of X, we have

$$\dim_c X = \sup_{U \in \mathcal{U}} \dim_c U.$$

In particular, $\dim_c U \leq \dim_c X$ for any open subspace U of X. Note that although a similar majoration holds for closed subspaces, it may be false for arbitrary subspaces.

Exercise 1.10.3. *Show that the cohomological dimension of an open subspace of $\mathbb{R}^n$ is equal to n. Deduce from this fact that if U and V are homeomorphic open subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ then $n = m$.*

Solution. Let us prove that $\dim_c \mathbb{R}^n \leq n$. The conclusion will follow since we know that

$$\mathrm{H}^n_c(B; \mathbb{Z}) = \mathbb{Z}$$

for any open ball B of $\mathbb{R}^n$.

Since

$$\frac{1}{2} + \frac{1}{\pi} \operatorname{arctg} x : \mathbb{R} \to]0, 1[$$

is a homeomorphism, $\mathbb{R}^n$ is homeomorphic to $]0,1[^n$. Since this last space is an open subspace of $[0,1]^n$, it is sufficient to show that $\dim_c [0,1]^n \le n$.

We will proceed by induction on n.

For $n = 1$, this follows from Exercise 1.6.7. Assuming the result is true for n, we prove it for $n+1$ by using the isomorphism

$$\mathrm{R}\Gamma([0,1]^{n+1};\mathcal{F}) \simeq \mathrm{R}\Gamma([0,1];p(\mathcal{F}))$$

(where $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a canonical projection) and the fibers formula for Rp.

The last part follows from the fact that cohomological dimension is clearly invariant by homeomorphism. □

Remark 1.10.4. It follows at once from the preceding exercise that the cohomological dimension of a differential manifold is equal to its usual dimension.

Exercise 1.10.5. *Let* Σ *be a finite simplicial complex. Show that the cohomological dimension of* $|\Sigma|$ *is equal to* $\dim \Sigma$.

Solution. We will proceed by induction on $\dim \Sigma$. For $\dim \Sigma = 0$, the result is obvious. To prove that the result is true for $\dim \Sigma = n+1$ if it is true for $\dim \Sigma \le n$, it is sufficient to use the excision distinguished triangle

$$\mathrm{R}\Gamma_c(|\Sigma_{n+1}| \setminus |\Sigma_n|;\mathcal{F}) \to \mathrm{R}\Gamma_c(|\Sigma_{n+1}|;\mathcal{F}) \to \mathrm{R}\Gamma_c(|\Sigma_n|;\mathcal{F}) \xrightarrow{+1}$$

together with the fact that $|\Sigma_{n+1}| \setminus |\Sigma_n|$ is a finite union of open cells of dimension $n+1$. □

Definition 1.10.6. The *reduced cohomology* $\tilde{\mathrm{H}}^{\cdot}(X;\mathbb{Z})$ of X with coefficient in $\mathbb{Z}$ is defined by setting

$$\tilde{\mathrm{H}}^k(X;\mathbb{Z}) = \begin{cases} \mathrm{H}^0(X;\mathbb{Z})/\mathbb{Z} & \text{if } k = 0, \\ \mathrm{H}^k(X;\mathbb{Z}) & \text{otherwise.} \end{cases}$$

A topological space X is *cohomologically locally connected* (clc for short) if, for any $x \in X$ and any neighborhood U of x, there is a neighborhood V of x included in U such that all the morphisms

$$\tilde{\mathrm{H}}^k(U;\mathbb{Z}) \to \tilde{\mathrm{H}}^k(V;\mathbb{Z})$$

are zero.

Examples 1.10.7. One checks directly that differential manifolds are clc spaces. With a little more work, one sees also that the same is true of compact polyhedra.

Proposition 1.10.8 (Borel-Wilder). *Assume X is a locally compact clc-space. Then, for any pair K, L of compact subsets of X such that $L \subset K^\circ$, all the restriction morphisms*

$$\mathrm{H}^k(K;\mathbb{Z}) \to \mathrm{H}^k(L;\mathbb{Z})$$

have finitely generated images.

Proof. We will proceed by increasing induction on k. Denote $\mathcal{L}$ the family of compact subsets L of K° which have a compact neighborhood $L' \subset K^\circ$ for which the image

$$r^k_{L'K} : \mathrm{H}^k(K;\mathbb{Z}) \to \mathrm{H}^k(L';\mathbb{Z})$$

is finitely generated. It is clear that any point $x \in K^\circ$ has neighborhood in $\mathcal{L}$. Hence, it is sufficient to show that if L_1, $L_2 \in \mathcal{L}$ then $L_1 \cup L_2 \in \mathcal{L}$. Choose compact neighborhoods L'_1, L'_2 of L_1, L_2 for which $r^k_{L'_1K}$ and $r^k_{L'_2K}$ have finitely generated images. Let L''_1, L''_2 be compact neighborhoods of L_1, L_2 such that $L''_1 \subset {L'_1}^\circ$, $L''_2 \subset {L'_2}^\circ$. Consider the diagram

$$\begin{array}{ccccc}
 & & \mathrm{H}^k(K;\mathbb{Z}) & \xrightarrow{\beta} & \mathrm{H}^k(K;\mathbb{Z}) \oplus \mathrm{H}^k(K;\mathbb{Z}) \\
 & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\epsilon} \\
\mathrm{H}^{k-1}(L'_1 \cap L'_2;\mathbb{Z}) & \xrightarrow{\alpha'} & \mathrm{H}^k(L'_1 \cup L'_2;\mathbb{Z}) & \xrightarrow{\beta'} & \mathrm{H}^k(L'_1;\mathbb{Z}) \oplus \mathrm{H}^k(L'_2;\mathbb{Z}) \\
\downarrow{\scriptstyle\gamma'} & & \downarrow{\scriptstyle\delta'} & & \\
\mathrm{H}^{k-1}(L''_1 \cap L''_2;\mathbb{Z}) & \xrightarrow{\alpha''} & \mathrm{H}^k(L''_1 \cup L''_2;\mathbb{Z}) & &
\end{array}$$

where the horizontal morphisms come from Mayer-Vietoris sequences and the vertical ones are restriction maps. We know that $\operatorname{Im}\epsilon$ is finitely generated. Since

$$\beta'(\operatorname{Im}\delta) \subset \operatorname{Im}\epsilon$$

we see that $\beta'(\operatorname{Im}\delta)$ is also finitely generated. Hence, so is $\operatorname{Im}\delta/(\operatorname{Im}\delta \cap \operatorname{Im}\alpha')$. Using the epimorphism

$$\delta'(\operatorname{Im}\delta/(\operatorname{Im}\delta \cap \operatorname{Im}\alpha')) \to \operatorname{Im}(\delta' \circ \delta)/\operatorname{Im}(\alpha'' \circ \gamma')$$

we see that $\operatorname{Im}(\delta' \circ \delta)/\operatorname{Im}(\alpha'' \circ \gamma')$ is finitely generated. Since the induction hypothesis shows that $\operatorname{Im}\gamma'$ is finitely generated, it follows that $\operatorname{Im}(\delta' \circ \delta)$ is finitely generated. This shows that $L_1 \cup L_2 \in \mathcal{L}$ and the conclusion follows. $\square$

Corollary 1.10.9. *Assume X is a compact clc space. Then, the abelian groups*

$$\mathrm{H}^k(X;\mathbb{Z})$$

are finitely generated. If, moreover, X has finite cohomological dimension, then the Euler-Poincaré characteristic

$$\chi(X) = \sum_{k\in\mathbb{Z}} \operatorname{rk} \mathrm{H}^k(X;\mathbb{Z})$$

is well-defined.

Remark 1.10.10. It follows from the preceding corollary that a compact differential manifold has a well-defined Euler-Poincaré characteristic. We will study it with more details in Chapter 2.

1.11 Poincaré-Verdier duality

Let $f : X \to Y$ be a continuous map between locally compact spaces.

Definition 1.11.1. A closed subset F of X is *f-proper* if the map

$$f_{|F} : F \to Y$$

is proper or in other words if $F \cap f^{-1}(K)$ is compact for every compact subset K of Y. Clearly, f-proper subsets of X form a family of supports.

Let $\mathcal{F}$ be a sheaf on X and let U be an open subset of Y. We set

$$f_!(\mathcal{F})(U) = \Gamma_{f-\text{proper}}(f^{-1}(U);\mathcal{F}).$$

One checks easily that $f_!(\mathcal{F})$ is a sheaf on Y. We call it the *direct image with proper supports of $\mathcal{F}$ by f.*

Proposition 1.11.2. *The functor*

$$f_! : \mathcal{S}hv(X) \to \mathcal{S}hv(Y)$$

is left exact and has a right derived functor

$$Rf_! : \mathcal{D}^+(\mathcal{S}hv(X)) \to \mathcal{D}^+(\mathcal{S}hv(Y))$$

which is computable by means of c-soft resolutions.

Remark 1.11.3. For the canonical map

$$a_X : X \to \{\mathrm{pt}\}$$

we see easily that

$$Ra_{X!}(\mathcal{F}) \simeq \mathrm{R}\Gamma_c(X;\mathcal{F}).$$

Proposition 1.11.4 (Fibers formula). *For any $\mathcal{F} \in \mathcal{D}^+(\mathcal{S}hv(X))$, we have a canonical isomorphism*

$$[Rf_!(\mathcal{F})]_y \simeq \mathrm{R}\Gamma_c(f^{-1}(y); \mathcal{F}).$$

Corollary 1.11.5. *The cohomological dimension of the functor $f_!$ is equal to*

$$\sup_{y \in Y}[\dim_c f^{-1}(y)]$$

Corollary 1.11.6 (Cartesian square formula). *Assume*

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle g'}\uparrow & \square & \uparrow{\scriptstyle g} \\ T & \xrightarrow[f']{} & Z \end{array}$$

is a cartesian square of locally compact spaces. Then, we have the canonical isomorphisms

$$g^{-1}f_! \simeq f'_!g'^{-1} \quad \text{and} \quad g^{-1}Rf_! \simeq Rf'_!g'^{-1}$$

Proposition 1.11.7. *Let $g : Y \to Z$ be another continuous map of locally compact spaces. Then, there are canonical isomorphisms*

$$(g \circ f)_! \simeq g_! \circ f_! \qquad \text{and} \qquad R(g \circ f)_! \simeq Rg_! \circ Rf_!.$$

Remark 1.11.8. Combining the preceding result with Remark 1.11.3, we see that

$$\mathrm{R}\Gamma_c(Y; Rf_!\mathcal{F}) \simeq \mathrm{R}\Gamma_c(X; \mathcal{F}).$$

A result which may be seen as a kind of Leray theorem with compact supports.

Theorem 1.11.9 (Poincaré-Verdier duality). *Assume $f_!$ has finite cohomological dimension (i.e. assume that there is $n \geq 0$ such that*

$$\mathrm{H}^k(Rf_!\mathcal{F}) = 0$$

for $k > n$ and any $\mathcal{F} \in \mathcal{S}hv(X)$). Then,

$$Rf_! : \mathcal{D}^+(\mathcal{S}hv(X)) \to \mathcal{D}^+(\mathcal{S}hv(Y))$$

has a right adjoint

$$f^! : \mathcal{D}^+(\mathcal{S}hv(Y)) \to \mathcal{D}^+(\mathcal{S}hv(X)).$$

In other words, there is a canonical functorial isomorphism

$$\operatorname{Hom}_{\mathcal{D}^+(\mathcal{S}hv(Y))}(Rf_!\mathcal{F},\mathcal{G}) \simeq \operatorname{Hom}_{\mathcal{D}^+(\mathcal{S}hv(X))}(\mathcal{F}, f^!\mathcal{G}).$$

Moreover, for $\mathcal{F} \in \mathcal{D}^-(\mathcal{S}hv(X))$ and $\mathcal{G} \in \mathcal{D}^+(\mathcal{S}hv(Y))$, there is a functorial isomorphism

$$\mathrm{RHom}\,(Rf_!\mathcal{F},\mathcal{G}) \simeq \mathrm{RHom}\,(\mathcal{F}, f^!\mathcal{G}).$$

Example 1.11.10. Let F be a closed subspace of X. Denote $\Gamma_F\mathcal{G}$ the abelian sheaf

$$U \mapsto \Gamma_{F\cap U}(U;\mathcal{G}).$$

Let $i : F \to X$ be the canonical inclusion of F in X. Then, one checks easily that $i_!$ is exact and that

$$\mathrm{RHom}\,(i_!\mathcal{F},\mathcal{G}) \simeq \mathrm{RHom}\,(\mathcal{F}, i^{-1}\mathrm{R}\Gamma_F\mathcal{G}).$$

In particular, there is a canonical functorial isomorphism

$$i^!\mathcal{G} \simeq i^{-1}\mathrm{R}\Gamma_F\mathcal{G}.$$

Corollary 1.11.11 (Absolute Poincaré duality). *Assume X is a finite dimensional locally compact space. Then, we have the canonical isomorphism*

$$\mathrm{RHom}\,(\mathrm{R}\Gamma_c(X;\mathbb{Z}_X),\mathbb{Z}) \simeq \mathrm{R}\Gamma(X;\omega_X)$$

where $\omega_X = a_X^!\mathbb{Z}$ is the dualizing complex of X.

Proof. Take $f = a_X$, $\mathcal{F} = \mathbb{Z}_X$, $\mathcal{G} = \mathbb{Z}$ in Theorem 1.11.9 and use the isomorphism

$$\mathrm{RHom}\,(\mathbb{Z}_X,\omega_X) \simeq \mathrm{R}\Gamma(X;\omega_X).$$

□

Proposition 1.11.12. *Let $g : Y \to Z$ be another continuous map of locally compact spaces. Then, the canonical isomorphism*

$$R(g\circ f)_! \simeq Rg_! \circ Rf_!$$

induces by adjunction the canonical isomorphism

$$(g\circ f)^! \simeq f^! \circ g^!.$$

Corollary 1.11.13 (Alexander duality). *Assume X has finite cohomological dimension and let F be a closed subset of X. Then, there is a canonical isomorphism*

$$\mathrm{RHom}\,(\mathrm{R}\Gamma_c(F;\mathbb{Z}),\mathbb{Z}) \simeq \mathrm{R}\Gamma_F(X;\omega_X).$$

Proof. It is clear that F is locally compact and has finite homological dimension. Moreover, thanks to Example 1.11.10 and Proposition 1.11.12, we have

$$\omega_F = a_F^! \mathbb{Z} \simeq i^! a_X^! \mathbb{Z} \simeq i^{-1} \mathrm{R}\Gamma_F \omega_X$$

where $i : F \to X$ is the canonical inclusion. Hence, the conclusion follows from Corollary 1.11.11 with X replaced by F. □

Proposition 1.11.14. *For $\mathbb{R}^n$ there is a canonical isomorphism*

$$\omega_{\mathbb{R}^n} \simeq \mathbb{Z}_{\mathbb{R}^n}[n].$$

Proof. Using Corollary 1.11.11, we see that

$$(\omega_{\mathbb{R}^n})_x \simeq \varinjlim_{\substack{U \ni x \\ U \text{ open ball}}} \mathrm{RHom}\,(\mathrm{R}\Gamma_c(U;\mathbb{Z}), \mathbb{Z}).$$

Thanks to Examples 1.8.4, we know that

$$\mathrm{R}\Gamma_c(U;\mathbb{Z}) \simeq \mathbb{Z}[-n].$$

Hence,

$$(\omega_{\mathbb{R}^n})_x \simeq \mathbb{Z}[n].$$

This shows in particular that

$$\mathrm{H}^k(\omega_{\mathbb{R}^n}) \simeq 0$$

for $k \neq -n$ and that the sheaf

$$\mathit{or}_{\mathbb{R}^n} = \mathrm{H}^{-n}(\omega_{\mathbb{R}^n})$$

has all its fibers canonically isomorphic to $\mathbb{Z}$. Moreover, it follows from Corollary 1.11.11 that for any open ball U of $\mathbb{R}^n$,

$$\mathrm{R}\Gamma(U; \mathit{or}_{\mathbb{R}^n}) \simeq \mathrm{RHom}\,(\mathrm{R}\Gamma_c(U;\mathbb{Z}_U)[n], \mathbb{Z}) \simeq \mathbb{Z}.$$

Using the fact (see Exercise 1.8.6) that the canonical diagram

$$\begin{array}{ccc} \mathrm{R}\Gamma_c(U;\mathbb{Z})[n] & \xrightarrow{\sim} & \mathbb{Z} \\ \downarrow & & \| \\ \mathrm{R}\Gamma_c(U';\mathbb{Z})[n] & \xrightarrow{\sim} & \mathbb{Z} \end{array}$$

is commutative if $U \subset U'$ are open balls of $\mathbb{R}^n$, we see that $\mathit{or}_{\mathbb{R}^n}$ is canonically isomorphic to $\mathbb{Z}_{\mathbb{R}^n}$. □

Corollary 1.11.15. *Let U (resp. F) be an open (resp. a closed) subset of $\mathbb{R}^n$. Then, we have the canonical isomorphisms*

$$\mathrm{RHom}\,(\mathrm{R}\Gamma_c(U;\mathbb{Z}),\mathbb{Z}) \simeq \mathrm{R}\Gamma(U;\mathbb{Z})[n]$$

and

$$\mathrm{RHom}\,(\mathrm{R}\Gamma_c(F;\mathbb{Z}),\mathbb{Z}) \simeq \mathrm{R}\Gamma_F(\mathbb{R}^n;\mathbb{Z})[n].$$

In particular, there are exact sequences of the form

$$0 \to \mathrm{Ext}^1(\mathrm{H}_c^{k+1}(U;\mathbb{Z}),\mathbb{Z}) \to \mathrm{H}^{n-k}(U;\mathbb{Z}) \to \mathrm{Hom}\,(\mathrm{H}_c^k(U;\mathbb{Z}),\mathbb{Z}) \to 0$$

and

$$0 \to \mathrm{Ext}^1(\mathrm{H}_c^{k+1}(F;\mathbb{Z}),\mathbb{Z}) \to \mathrm{H}_F^{n-k}(\mathbb{R}^n;\mathbb{Z}) \to \mathrm{Hom}\,(\mathrm{H}_c^k(F;\mathbb{Z}),\mathbb{Z}) \to 0.$$

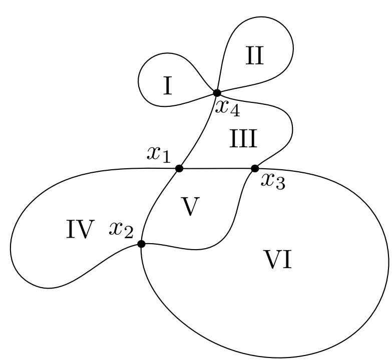
Proof. The first part follows from Corollary 1.11.11 and Corollary 1.11.13 combined with the preceding proposition.

As for the second part, it follows from the fact that for any complex $C^\cdot$ of abelian groups we have an exact sequence of the form

$$0 \to \mathrm{Ext}^1(\mathrm{H}^{k+1}(C^\cdot),\mathbb{Z}) \to \mathrm{H}^{-k}(\mathrm{RHom}\,(C^\cdot,\mathbb{Z})) \to \mathrm{Hom}\,(\mathrm{H}^k(C^\cdot),\mathbb{Z}) \to 0$$

for any $k \in \mathbb{Z}$. □

Exercise 1.11.16. *Let $f : S_1 \to \mathbb{R}^2$ be a continuous map and set $C = f(S_1)$. We define the multiplicity of $x \in C$ to be $\mu_x = \#f^{-1}(x)$. A point $x \in C$ is simple if $\mu_x = 1$ and multiple otherwise. Assume C has a finite number of multiple points each of which has finite multiplicity. Compute the cohomology of C and $\mathbb{R}^2 \setminus C$. In particular, show that the number of bounded connected components of $\mathbb{R}^2 \setminus C$ is $1 + \sum_{x\in C}(\mu_x - 1)$ generalizing in this way the well-known theorem on Jordan curves. For example in*



we have

$$\mu_{x_1} = \mu_{x_2} = \mu_{x_3} = 2, \qquad \mu_{x_4} = 3$$

and

$$1 + \sum_{x \in C} (\mu_x - 1) = 1 + 1 + 1 + 1 + 2 = 6.$$

Solution. Let $g : S_1 \to C$ be the map induced by f. We have

$$Rg(\mathbb{Z}_{S_1})_x \simeq \mathrm{R}\Gamma(g^{-1}(x); \mathbb{Z}) \simeq \mathbb{Z}^{\mu_x} \tag{*}$$

for any $x \in C$. In particular, $Rg(\mathbb{Z}_{S_1}) \simeq g(\mathbb{Z}_{S_1})$. Consider the canonical morphism $\mathbb{Z}_C \to g(\mathbb{Z}_{S_1})$. Since g is surjective, it is a monomorphism. Moreover, (*) shows that its cokernel is supported by the set of multiple points of C. Therefore, we get the exact sequence of sheaves

$$0 \to \mathbb{Z}_C \to g(\mathbb{Z}_{S_1}) \to \bigoplus_{\substack{x \in C \\ \mu_x > 1}} \mathbb{Z}_{\{x\}}^{\mu_x - 1} \to 0$$

where $\mathbb{Z}_{\{x\}}$ is the direct image of the constant sheaf on $\{x\}$ in C. Applying $\mathrm{R}\Gamma(C; \cdot)$ we get the distinguished triangle

$$0 \to \mathrm{R}\Gamma(C; \mathbb{Z}_C) \to \mathrm{R}\Gamma(C; Rg(\mathbb{Z}_{S_1})) \to \mathrm{R}\Gamma(C; \bigoplus_{\substack{x \in C \\ \mu_x > 1}} \mathbb{Z}_{\{x\}}^{\mu_x - 1}) \xrightarrow{+1}$$

Moreover,

$$\mathrm{R}\Gamma(C; Rg(\mathbb{Z}_{S_1})) \simeq \mathrm{R}\Gamma(S_1; \mathbb{Z}_{S_1}) \quad \text{and} \quad \mathrm{R}\Gamma(C; \bigoplus_{\substack{x \in C \\ \mu_x > 1}} \mathbb{Z}_{\{x\}}^{\mu_x - 1}) \simeq \bigoplus_{\substack{x \in C \\ \mu_x > 1}} \mathbb{Z}^{\mu_x - 1}.$$

Taking cohomology and setting

$$\nu = \sum_{\substack{x \in C \\ \mu_x > 1}} (\mu_x - 1),$$

we get the exact sequence

$$0 \to \mathrm{H}^0(C; \mathbb{Z}_C) \to \mathrm{H}^0(S_1; \mathbb{Z}_{S_1}) \to \mathbb{Z}^\nu \to \mathrm{H}^1(C; \mathbb{Z}_C) \to \mathrm{H}^1(S_1; \mathbb{Z}_{S_1}) \to 0$$

and the isomorphisms

$$\mathrm{H}^k(C; \mathbb{Z}_C) \simeq \mathrm{H}^k(S_1; \mathbb{Z}_{S_1}) \simeq 0$$

for $k > 1$. Since S_1 is connected, so is C and $\mathrm{H}^0(C; \mathbb{Z}_C) \simeq \mathrm{H}^0(S_1; \mathbb{Z}_{S_1}) \simeq \mathbb{Z}$. Through these isomorphisms, the first morphism appears as $\mathrm{id} : \mathbb{Z} \to \mathbb{Z}$. Hence, using the fact that $\mathrm{H}^1(S_1; \mathbb{Z}_{S_1}) \simeq \mathbb{Z}$, we get the exact sequence

$$0 \to \mathbb{Z}^\nu \to \mathrm{H}^1(C; \mathbb{Z}_C) \to \mathbb{Z} \to 0.$$

The last group being projective, we get

$$\mathrm{H}^1(C;\mathbb{Z}_C) \simeq \mathbb{Z}^{\nu+1}.$$

The cohomology table for C is thus

$$\mathrm{H}^k(C;\mathbb{Z}_C) \simeq \begin{cases} \mathbb{Z} & \text{if } k=0, \\ \mathbb{Z}^{1+\nu} & \text{if } k=1, \\ 0 & \text{otherwise.} \end{cases}$$

Since all these groups are free, Corollary 1.11.15 shows that

$$\mathrm{H}^{2-k}_C(\mathbb{R}^2;\mathbb{Z}_{\mathbb{R}^2}) \simeq \mathrm{Hom}\,(\mathrm{H}^k(C;\mathbb{Z}),\mathbb{Z}).$$

Therefore,

$$\mathrm{H}^{k}_C(\mathbb{R}^2;\mathbb{Z}_{\mathbb{R}^2}) \simeq \begin{cases} \mathbb{Z}^{1+\nu} & \text{if } k=1, \\ \mathbb{Z} & \text{if } k=2, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the excision distinguished triangle

$$\mathrm{R}\Gamma_C(\mathbb{R}^2;\mathbb{Z}) \to \mathrm{R}\Gamma(\mathbb{R}^2;\mathbb{Z}) \to \mathrm{R}\Gamma(\mathbb{R}^2\setminus C;\mathbb{Z}) \xrightarrow{+1}$$

Taking cohomology and using the fact that $\mathrm{H}^k(\mathbb{R}^2;\mathbb{Z}) \simeq 0$ for $k>0$, we get the exact sequence

$$0 \to \mathrm{H}^0_C(\mathbb{R}^2;\mathbb{Z}) \to \mathrm{H}^0(\mathbb{R}^2;\mathbb{Z}) \to \mathrm{H}^0(\mathbb{R}^2\setminus C;\mathbb{Z}) \to \mathrm{H}^1_C(\mathbb{R}^2;\mathbb{Z}) \to 0$$

and the isomorphisms

$$\mathrm{H}^k(\mathbb{R}^2\setminus C;\mathbb{Z}) \simeq \mathrm{H}^{k+1}_C(\mathbb{R}^2;\mathbb{Z})$$

for $k>0$. Since $\mathbb{R}^2$ is connected, a locally constant function in $\mathbb{R}^2$ is constant and we have $\mathrm{H}^0(\mathbb{R}^2;\mathbb{Z}) \simeq \mathbb{Z}$ and $\mathrm{H}^0_C(\mathbb{R}^2;\mathbb{Z}) \simeq 0$. It follows that the sequence

$$0 \to \mathbb{Z} \to \mathrm{H}^0(\mathbb{R}^2\setminus C;\mathbb{Z}) \to \mathbb{Z}^{1+\nu} \to 0$$

is exact. From these results we deduce that

$$\mathrm{H}^k(\mathbb{R}^2\setminus C;\mathbb{Z}) \simeq \begin{cases} \mathbb{Z}^{2+\nu} & \text{if } k=0, \\ \mathbb{Z} & \text{if } k=1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the number of connected components of $\mathbb{R}^2\setminus C$ is $2+\nu$. Since C is compact, $\mathbb{R}^2\setminus C$ has exactly one non bounded connected component. It follows that the number of bounded connected components of $\mathbb{R}^2 \setminus C$ is $1+\nu$. Moreover, our method of proof shows also that

$$\mathrm{H}^1(\mathbb{R}^2 \setminus C; \mathbb{Z}) \to \mathrm{H}^1(\mathbb{R}^2 \setminus \overline{B}; \mathbb{Z})$$

is an isomorphism if $\overline{B}$ is a closed ball of $\mathbb{R}^2$ containing C. It follows that, for any bounded connected component U of $\mathbb{R}^2 \setminus C$, we have

$$\mathrm{H}^k(U; \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ 0 & \text{otherwise} \end{cases}$$

and that

$$\mathrm{H}^k(U; \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } k = 0, 1, \\ 0 & \text{otherwise} \end{cases}$$

if U is the non bounded connected component of $\mathbb{R}^2 \setminus C$. □

1.12 Borel-Moore homology

From the point of view of sheaf theory, cohomology is more natural than homology. However, to make the link with classical homology theories, it is convenient to introduce the following kind of co-cohomology.

Definition 1.12.1. Let Φ be a family of supports of X. We define the *Borel-Moore homology* $\mathrm{H}_k^{\Phi}(X; \mathbb{Z})$ of X with integer coefficients and supports in Φ by setting

$$\mathrm{H}_k^{\Phi}(X; \mathbb{Z}) = \mathrm{H}_{\Phi}^{-k}\mathrm{R}\Gamma(X; \omega_X).$$

Remark 1.12.2. Note that thanks to Corollary 1.11.11, we have a canonical epimorphism

$$\mathrm{H}_k(X; \mathbb{Z}) \to \mathrm{Hom}\,(\mathrm{H}_c^k(X; \mathbb{Z}), \mathbb{Z})$$

which becomes an isomorphism when $\mathrm{H}_c^{k+1}(X; \mathbb{Z})$ has no torsion.

Proposition 1.12.3. *If X is a Homologically Locally Connected space in the sense of singular homology (HLC for short), then there are canonical isomorphisms*

$$\mathrm{H}_k^c(X; \mathbb{Z}) \simeq \mathrm{SH}_k(X; \mathbb{Z}) \qquad (k \in \mathbb{N})$$

where $\mathrm{SH}_k(X; \mathbb{Z})$ denotes the singular homology with integer coefficients of X.

Corollary 1.12.4. *Assume X is a HLC space and K is a compact subset of X. Then, there are canonical isomorphisms*

$$\mathrm{H}^{-k}\mathrm{R}\Gamma(K; \omega_X) \simeq \mathrm{SH}_k(X, X \setminus K; \mathbb{Z}).$$

Proof. This follows from the preceding proposition combined with the distinguished triangle

$$\mathrm{R}\Gamma_c(X \setminus K; \omega_X) \to \mathrm{R}\Gamma_c(X; \omega_X) \to \mathrm{R}\Gamma(K; \omega_X) \xrightarrow{+1}$$

□

Proposition 1.12.5. *Let $f : X \to Y$ be a continuous map and assume X, Y are locally compact spaces of finite cohomological dimension. Then, there is a canonical morphism*

$$Rf_! \omega_X \to \omega_Y.$$

This morphism induces a morphism

$$f_* : \mathrm{H}^c_\cdot(X; \mathbb{Z}) \to \mathrm{H}^c_\cdot(Y; \mathbb{Z})$$

which is compatible with the usual push-forward morphism

$$f_* : \mathrm{SH}_\cdot(X; \mathbb{Z}) \to \mathrm{SH}_\cdot(Y; \mathbb{Z}).$$

Proof. We have

$$f^! \omega_Y \simeq f^! a_Y^! \mathbb{Z} \simeq a_X^! \mathbb{Z} \simeq \omega_X.$$

Thanks to Poincaré-Verdier duality,

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{S}hv(Y))}(Rf_! \omega_X, \omega_Y) \simeq \mathrm{Hom}_{\mathcal{D}(\mathcal{S}hv(X))}(\omega_X, f^! \omega_Y).$$

Hence, to the isomorphism $\omega_X \simeq f^! \omega_Y$ corresponds a canonical morphism

$$Rf_! \omega_X \to \omega_Y.$$

Applying $\mathrm{R}\Gamma_c(Y; \cdot)$ to this morphism, we get a morphism

$$\mathrm{R}\Gamma_c(X; \omega_X) \to \mathrm{R}\Gamma_c(Y; \omega_Y)$$

which induces f_* at the level of cohomology. For the link with singular homology, we refer to standard texts. □

Remark 1.12.6. For $f = a_X$, we see that

$$(a_X)_* : \mathrm{H}^c_0(X; \mathbb{Z}) \to \mathbb{Z}$$

corresponds to the classical augmentation

$$\# : \mathrm{SH}_0(X; \mathbb{Z}) \to \mathbb{Z}.$$

which associates to a singular 0-cycle $c = \sum_{j=1}^J m_j [x_j]$ the number $\#c = \sum_{j=1}^J m_j$. This is why we will often use $\#$ as a shorthand notation for $(a_X)_*$.

1.13 Products in cohomology and homology

Definition 1.13.1. Let $\mathcal{F}$, $\mathcal{G}$ be two abelian sheaves on the topological space X. We define the *tensor product* of $\mathcal{F}$ and $\mathcal{G}$ to be the abelian sheaf associated to the presheaf

$$U \mapsto \mathcal{F}(U) \otimes \mathcal{G}(U).$$

Let Φ and Ψ be families of supports on X. Since $F \cap G \in \Phi \cap \Psi$ for any $F \in \Phi$ and any $G \in \Psi$, we have a canonical morphism

$$\smile : \Gamma_\Phi(X;\mathcal{F}) \otimes \Gamma_\Psi(X;\mathcal{G}) \to \Gamma_{\Phi\cap\Psi}(X;\mathcal{F}\otimes\mathcal{G}).$$

Proposition 1.13.2. *The functor*

$$\otimes : \mathcal{S}hv(X) \times \mathcal{S}hv(X) \to \mathcal{S}hv(X)$$

is right exact, left derivable and has finite homological dimension. Moreover, if Φ and Ψ are families of supports on X, we have a canonical morphism

$$\smile : \mathrm{R}\Gamma_\Phi(X;\mathcal{F}) \otimes^L \mathrm{R}\Gamma_\Psi(X;\mathcal{G}) \to \mathrm{R}\Gamma_{\Phi\cap\Psi}(X;\mathcal{F}\otimes^L\mathcal{G})$$

which induces the generalized cup-products

$$\smile : \mathrm{H}^k_\Phi(X;\mathcal{F}) \otimes \mathrm{H}^l_\Psi(X;\mathcal{G}) \to \mathrm{H}^{k+l}_{\Phi\cap\Psi}(X;\mathcal{F}\otimes^L\mathcal{G})$$

at the level of cohomology.

Remark 1.13.3. For $\mathcal{F} = \mathcal{G} = \mathbb{Z}_X$, we have $\mathcal{F}\otimes^L\mathcal{G} \simeq \mathbb{Z}_X$ and one can show that the morphisms

$$\smile : \mathrm{H}^k_\Phi(X;\mathbb{Z}_X) \otimes \mathrm{H}^l_\Psi(X;\mathbb{Z}_X) \to \mathrm{H}^{k+l}_{\Phi\cap\Psi}(X;\mathbb{Z}_X)$$

given by the preceding proposition coincide with the classical cup-products. On can also recover the usual formulas

$$(c^k \smile c^l) \smile c^m = c^k \smile (c^l \smile c^m) \qquad \text{and} \qquad c^k \smile c^l = (-1)^{kl} c^l \smile c^k.$$

For $\mathcal{F} = \mathbb{Z}_X$ and $\mathcal{G} = \omega_X$, we have $\mathcal{F}\otimes^L\mathcal{G} = \omega_X$ and we get the morphisms

$$\smile : \mathrm{H}^k_\Phi(X;\mathbb{Z}_X) \otimes \mathrm{H}^{-l}_\Psi(X;\omega_X) \to \mathrm{H}^{k-l}_{\Phi\cap\Psi}(X;\omega_X).$$

Using the equality $\mathrm{H}^{-m}_\Psi(X;\omega_X) = \mathrm{H}^\Psi_m(X;\mathbb{Z}_X)$, we recover the classical cap-products

$$\frown : \mathrm{H}^k_\Phi(X;\mathbb{Z}_X) \otimes \mathrm{H}^\psi_l(X;\mathbb{Z}_X) \to \mathrm{H}^{\Phi\cap\Psi}_{l-k}(X;\mathbb{Z}_X).$$

Thanks to the associativity of the generalized cup-products, we also recover the cup-cap associativity formula

$$(c^k \smile c^l) \frown c_m = c^k \frown (c^l \frown c_m).$$

If the elements of $\Phi \cap \Psi$ are compact, we also get a pairing

$$\langle \cdot, \cdot \rangle : \mathrm{H}^k_\Phi(X;\mathbb{Z}) \otimes \mathrm{H}^\Psi_k(X;\mathbb{Z}) \to \mathbb{Z}$$

by composing

$$\frown : \mathrm{H}^k_\Phi(X;\mathbb{Z}) \otimes \mathrm{H}^\Psi_k(X;\mathbb{Z}) \to \mathrm{H}^{\Phi\cap\Psi}_0(X;\mathbb{Z})$$

with

$$\# : \mathrm{H}^c_0(X;\mathbb{Z}) \to \mathbb{Z}.$$

This pairing is a natural generalization of the classical pairing between homology and cohomology. Thanks to the cup-cap associativity formula, we have

$$\left\langle c^k \smile c^l, c_{k+l} \right\rangle = \left\langle c^k, c^l \frown c_{k+l} \right\rangle.$$

Proposition 1.13.4 (Projection formula). *Let $f : X \to Y$ be a continuous map between locally compact spaces. Then, for any $\mathcal{F} \in \mathcal{D}^+(\mathcal{S}hv(X))$ and any $\mathcal{G} \in \mathcal{D}^+(\mathcal{S}hv(Y))$, there is a canonical isomorphism*

$$\mathcal{G} \otimes^L Rf_!\mathcal{F} \xrightarrow{\sim} Rf_!(f^{-1}\mathcal{G} \otimes^L \mathcal{F}).$$

Corollary 1.13.5 (Universal coefficient formula). *Let X be a locally compact space. Then, for any $M \in \mathcal{D}^+(\mathcal{A}b)$, we have the canonical isomorphism*

$$\mathrm{R}\Gamma_c(X;M) \simeq M \otimes^L \mathrm{R}\Gamma_c(X;\mathbb{Z}).$$

Remark 1.13.6. Thanks to the preceding proposition, one can prove that if $f : X \to Y$ is a morphism of locally compact spaces we have

$$f_*(f^*(c^k) \frown c_l) = c^k \frown f_*(c_l)$$

for any $c^k \in \mathrm{H}^k(Y;\mathbb{Z})$ and any $c_l \in \mathrm{H}^c_l(X;\mathbb{Z})$. In particular, we get

$$\left\langle f^*(c^k), c_l \right\rangle = \left\langle c^k, f_*(c_l) \right\rangle$$

if $k = l$.

Definition 1.13.7. Let X, Y be topological spaces and let $\mathcal{F}$ (resp. $\mathcal{G}$) be a sheaf on X (resp. Y). Denote p_X, p_Y the canonical projections of $X \times Y$

on X and Y. We define the exterior tensor product $\mathcal{F} \boxtimes \mathcal{G}$ of $\mathcal{F}$ and $\mathcal{G}$ by the formula

$$\mathcal{F} \boxtimes \mathcal{G} = p_X^{-1}\mathcal{F} \otimes p_Y^{-1}\mathcal{G}.$$

Let Φ and Ψ be families of supports on X and Y. Denote $\Phi \times \Psi$ the family of supports on $X \times Y$ formed by the closed subsets of the products of the form $F \times G$ with $F \in \Phi$ and $G \in \Psi$. By definition of $\mathcal{F} \boxtimes \mathcal{G}$, we get a canonical morphism

$$\times : \Gamma_\Phi(X;\mathcal{F}) \otimes \Gamma_\Psi(Y;\mathcal{G}) \to \Gamma_{\Phi\times\Psi}(X \times Y;\mathcal{F} \boxtimes \mathcal{G})$$

Proposition 1.13.8. *Assume X, Y are topological spaces. Then, the functor*

$$\boxtimes : \mathcal{S}hv(X) \times \mathcal{S}hv(Y) \to \mathcal{S}hv(X \times Y)$$

is right exact and left derivable. It has finite homological dimension and if Φ and Ψ are families of supports on X and Y, we have a canonical morphism

$$\mathrm{R}\Gamma_\Phi(X;\mathcal{F}) \otimes^L \mathrm{R}\Gamma_\Psi(Y;\mathcal{G}) \to \mathrm{R}\Gamma_{\Phi\times\Psi}(X \times Y;\mathcal{F} \boxtimes^L \mathcal{G})$$

which induces the generalized cross-products

$$\times : \mathrm{H}^k_\Phi(X;\mathcal{F}) \otimes \mathrm{H}^l_\Psi(Y;\mathcal{G}) \to \mathrm{H}^{k+l}_{\Phi\times\Psi}(X \times Y;\mathcal{F} \boxtimes \mathcal{G})$$

at the level of cohomology.

Remark 1.13.9. If $\mathcal{F} = \mathbb{Z}_X$ and $\mathcal{G} = \mathbb{Z}_Y$, we get $\mathcal{F} \boxtimes \mathcal{G} \simeq \mathbb{Z}_{X\times Y}$ and the generalized cross-products

$$\times : \mathrm{H}^k_\Phi(X;\mathbb{Z}) \otimes \mathrm{H}^l_\Psi(Y;\mathbb{Z}) \to \mathrm{H}^{k+l}_{\Phi\times\Psi}(X \times Y;\mathbb{Z})$$

corresponds to the classical cross-products in cohomology. Note that this is not really a new operation since

$$c^k \times c^l = p_X^*(c^k) \smile p_Y^*(c^l).$$

So, most of the properties of cross-products in cohomology may be deduced from this formula. In particular, if $f : X \to X'$ and $g : Y \to Y'$ are continuous maps, we have

$$(f \times g)^*(c^k \times c^l) = f^*(c^k) \times g^*(c_l).$$

Proposition 1.13.10 (Künneth theorem for cohomology).
Assume X, Y are locally compact spaces. Then, the canonical morphism

$$\mathrm{R}\Gamma_c(X;\mathcal{F}) \otimes^L \mathrm{R}\Gamma_c(Y;\mathcal{G}) \to \mathrm{R}\Gamma_c(X \times Y;\mathcal{F} \boxtimes^L \mathcal{G})$$

is an isomorphism for any $\mathcal{F}$ in $\mathcal{D}^+(\mathcal{S}hv(X))$ and any $\mathcal{G}$ in $\mathcal{D}^+(\mathcal{S}hv(Y))$.

Proof. This follows directly from the projection formula and the cartesian square formula. ☐

Lemma 1.13.11. *Assume X and Y are locally compact spaces. Denote*

$$p_Y : X \times Y \to Y$$

the second projection. Then, there is a functorial canonical morphism

$$\omega_X \boxtimes^L \mathcal{G} \to p_Y^! \mathcal{G}$$

for $\mathcal{G} \in \mathcal{D}^+(\mathcal{S}hv(Y))$. This morphism becomes an isomorphism if X is a clc space.

Proof. We have the chain of morphisms

$$\begin{aligned} Rp_{Y!}(\omega_X \boxtimes^L \mathcal{G}) &\simeq Rp_{Y!}(p_X^{-1}\omega_X \otimes^L p_Y^{-1}\mathcal{G}) && (1)\\ &\simeq Rp_{Y!}(p_X^{-1}\omega_X) \otimes^L \mathcal{G} && (2)\\ &\simeq a_Y^{-1} Ra_{X!}\omega_X \otimes^L \mathcal{G} && (3) \end{aligned}$$

where (1) comes from the definition of $\boxtimes$, (2) follows from the projection formula and (3) from the cartesian square formula. Using the canonical morphism

$$Ra_{X!}\omega_X \to \mathbb{Z},$$

we get a canonical morphism

$$Rp_{Y!}(\omega_X \boxtimes^L \mathcal{G}) \to \mathcal{G}.$$

By adjunction, this gives us the requested morphism

$$\omega_X \boxtimes^L \mathcal{G} \to p_Y^! \mathcal{G}. \qquad (*)$$

At the level of sections, these morphisms may be visualized as follows. Let U, V be open subsets of X and Y. Then, on one hand, we have

$$\begin{aligned} \mathrm{R}\Gamma(U \times V, p_Y^! \mathcal{G}) &\simeq \mathrm{RHom}\,(\mathbb{Z}_{U\times V}, p_Y^! \mathcal{G}_{|V})\\ &\simeq \mathrm{RHom}\,(Rp_{V!}\mathbb{Z}_{U\times V}, \mathcal{G}_{|V})\\ &\simeq \mathrm{RHom}\,(\mathrm{R}\Gamma_c(U; \mathbb{Z}_U), \mathrm{R}\Gamma(V; \mathcal{G})). \end{aligned}$$

On the other hand,

$$\mathrm{R}\Gamma(U; \omega_X) \otimes^L \mathrm{R}\Gamma(V; \mathcal{G}) \simeq \mathrm{RHom}\,(\mathrm{R}\Gamma_c(U; \mathbb{Z}_U), \mathbb{Z}) \otimes^L \mathrm{R}\Gamma(V; \mathcal{G})$$

and the canonical morphism

$$\mathrm{R}\Gamma(U;\omega_X)\otimes^L \mathrm{R}\Gamma(V;\mathcal{G})\to \mathrm{R}\Gamma(U\times V;p_Y^!\mathcal{G})$$

induced by (*) corresponds to the canonical morphism

$$\mathrm{RHom}\,(\mathrm{R}\Gamma_c(U;\mathbb{Z}_U),\mathbb{Z})\otimes^L \mathrm{R}\Gamma(V;\mathcal{G})\to \mathrm{RHom}\,(\mathrm{R}\Gamma_c(U;\mathbb{Z}),\mathrm{R}\Gamma(V;\mathcal{G})).$$

The last part of the result follows by taking limits and cohomology and using the fact that

$$\mathrm{RHom}\,(C^\cdot,\mathbb{Z})\otimes^L P^\cdot \simeq \mathrm{RHom}\,(C^\cdot,P^\cdot)$$

if $C^\cdot$ is a bounded complex with finitely generated cohomology. □

Proposition 1.13.12. *Let X and Y be locally compact topological spaces. Then, there is a canonical morphism*

$$\omega_X\boxtimes^L\omega_Y\longrightarrow\omega_{X\times Y}. \tag{1}$$

If Φ and Ψ be families of supports on X and Y, this morphism induces a canonical morphism

$$\mathrm{R}\Gamma_\Phi(X;\omega_X)\otimes^L \mathrm{R}\Gamma_\Psi(Y;\omega_Y)\to \mathrm{R}\Gamma_{\Phi\times\Psi}(\omega_{X\times Y}) \tag{2}$$

and, hence, cross-products in homology

$$\times:\mathrm{H}_k^\Phi(X;\mathbb{Z})\otimes\mathrm{H}_l^\Psi(Y;\mathbb{Z})\to\mathrm{H}_{k+l}^{\Phi\times\Psi}(X\times Y;\mathbb{Z}).$$

Moreover, if X or Y is a clc space and Φ (resp. Ψ) is the family of all compact subsets of X (resp. Y), then both (1) and (2) are isomorphisms.

Proof. This follows directly from the preceding lemma and the fact that $p_Y^!\omega_Y=\omega_{X\times Y}$. □

Remark 1.13.13. A link between the cross-products in homology and cohomology is given by the formula

$$(c^p\times c^q)\frown(c_r\times c_s)=(-1)^{p(s-q)}(c^p\frown c_r)\times(c^q\frown c_s)$$

which entails the formula

$$\left\langle c^k\times c^l,c_k\times c_l\right\rangle=\left\langle c^k,c_k\right\rangle\left\langle c^l,c_l\right\rangle.$$

Note also that homology cross-products are compatible with push-forwards. Namely, if $f:X\to X'$ and $g:Y\to Y'$ are morphisms of locally compact spaces, then

$$(f\times g)_*(c_k\times c_l)=f_*(c_k)\times g_*(c_l).$$

1.14 Cohomology of topological manifolds

Definition 1.14.1. A *topological manifold of dimension* n is a Haussdorf topological space which is locally isomorphic to $\mathbb{R}^n$.

Proposition 1.14.2. *A topological manifold X of dimension n is a clc locally compact space. Its cohomological dimension is n, ω_X is concentrated in degree $-n$ and $\mathrm{H}^{-n}(\omega_X)$ is a locally constant sheaf with fiber $\mathbb{Z}$.*

Proof. This follows directly from the definition and Proposition 1.11.14. ☐

Definition 1.14.3. We define the orientation sheaf or_X of X by setting

$$or_X = \mathrm{H}^{-n}(\omega_X).$$

The manifold X is *orientable* if and only if the sheaf or_X is constant. In such a case, an *orientation* of X is an isomorphism

$$\mathbb{Z}_X \xrightarrow{\sim} or_X\,.$$

The manifold X endowed with an orientation forms an *oriented manifold.* The orientation class of the oriented manifold X is the section

$$\mu_X \in \Gamma(X; or_X)$$

image of the section

$$1_X \in \Gamma(X; \mathbb{Z}_X)$$

by the orientation of X.

Proposition 1.14.4. *Assume X is a topological manifold. Then,*

$$\Gamma(K; or_X) \simeq \mathrm{SH}_n(X, X \setminus K; \mathbb{Z})$$

canonically for any compact subset K of X. In particular, the notions related to orientability considered above are compatible with the ones considered in singular homology.

Proof. As a matter of fact, $or_X \simeq \omega_X[-n]$ and

$$\Gamma(K; or_X) \simeq \mathrm{H}^0\mathrm{R}\Gamma(K; \omega_X[-n]) \simeq \mathrm{H}^{-n}\mathrm{R}\Gamma(K; \omega_X)$$

and the announced isomorphism follows from Corollary 1.12.4. Recall that an orientation of X from the point of view of singular homology corresponds to the data of a generator $\mu_x \in \mathrm{SH}_n(X, X \setminus \{x\}; \mathbb{Z})$ for any $x \in X$ in such a way that for any $x_0 \in X$ there is a neighborhood K of x_0 and

$\mu_K \in \mathrm{SH}_n(X, X\backslash K; \mathbb{Z})$ such that μ_x is the image of μ_K in $\mathrm{SH}_n(X, X\backslash\{x\}; \mathbb{Z})$ for any $x \in K$. Using the isomorphism

$$\Gamma(K; or_X) \simeq \mathrm{SH}_n(X, X \setminus K; \mathbb{Z})$$

one sees easily that the family $(\mu_x)_{x\in X}$ corresponds to a section μ of or_X on X which generates $(or_X)_x$ for any $x \in X$ and the conclusion follows. $\square$

Remark 1.14.5.
(a) If X is orientable and connected, we have

$$\mathrm{Hom}\,(\mathbb{Z}_X, \mathbb{Z}_X) \simeq \Gamma(X; \mathbb{Z}_X) \simeq \mathbb{Z}.$$

It follows that the sheaf $\mathbb{Z}_X$ has only two automorphisms ($\pm\,\mathrm{id}$). Therefore, X has exactly two orientations. If μ_X is the class of one of them, $-\mu_X$ is the class of the other.

(b) Any open subset U of a topological manifold X is a topological manifold. If μ_X is an orientation class of X, $\mu_{X_{|U}}$ is an orientation class for U.

(c) We may restate Proposition 1.11.14 by stating that the topological manifold $\mathbb{R}^n$ is canonically oriented. We will denote $\mu_{\mathbb{R}^n}$ the corresponding orientation class.

(d) Since $\omega_X = or_X[n]$, we have

$$\Gamma(U; or_X) = \mathrm{H}^{-n}(U; \omega_X).$$

Thanks to Corollary 1.11.11, we get that

$$\Gamma(U; or_X) = \mathrm{H}^{-n}(\mathrm{RHom}\,(\mathrm{R}\Gamma_c(U; \mathbb{Z}_X), \mathbb{Z})).$$

Since $\mathrm{H}^{n+1}_c(U; \mathbb{Z}_X) \simeq 0$, we obtain a canonical isomorphism

$$\Gamma(U; or_X) \simeq \mathrm{Hom}\,(\mathrm{H}^n_c(U; \mathbb{Z}_X), \mathbb{Z}).$$

This provides a more explicit way to view the sheaf or_X.

Definition 1.14.6. A homeomorphism $\varphi : X \to Y$ of oriented topological manifolds is oriented if the orientation class of X corresponds to the orientation class of Y through the canonical isomorphism

$$\varphi^{-1}\, or_Y \simeq or_X\, .$$

Lemma 1.14.7. *A diffeomorphism $\varphi : U \to V$ between open subsets of $\mathbb{R}^n$ is oriented if and only if its Jacobian J_φ is strictly positive.*

Proof. Let x_0 and y_0 be points of U and V. Denote μ_U and μ_V the canonical orientation classes of U and V and denote ν_U the image of μ_U by the canonical isomorphism $\varphi^{-1}\, or_V \simeq or_U$. Let B_U (resp. B_V) be an open ball of U (resp. V) containing x_0 (resp. y_0). Assume that $\varphi(B_U) \subset B_V$. Using part (d) of Remark 1.14.5, we see that $\nu_{U_{|B_U}} = \pm\mu_{U_{|B_U}}$ according to the fact that the diagram

$$\begin{tikzcd}
\mathrm{H}^n_c(B_U;\mathbb{Z}) \arrow[r, "\int_U"] & \mathbb{Z} \\
\mathrm{H}^n_c(\varphi(B_U);\mathbb{Z}) \arrow[u, "\varphi^*", "\wr"'] \arrow[d, "i"'] & \\
\mathrm{H}^n_c(B_V;\mathbb{Z}) \arrow[uur, "\int_V"'] &
\end{tikzcd}$$

commutes or anticommutes. Using Exercise 1.8.6, we know that if

$$v_{\varphi(B_U)} \in \mathrm{H}^n_c(\varphi(B_U);\mathbb{Z})$$

has integral 1 then so has $i(v_{\varphi(B_U)})$. Let m be an integer such that

$$\varphi^*(v_{\varphi(B_U)}) = m v_U$$

where $v_U \in \mathrm{H}^n_c(B_U;\mathbb{Z})$ is a class with integral 1. On one hand, we have

$$\int \varphi^*(v_{\varphi(B_U)}) = m.$$

On the other hand, representing the classes by means of de Rham complexes, we get

$$\int \varphi^*(v_{\varphi(B_U)}) = \pm \int v_{\varphi(B_U)} = \pm 1$$

according to the fact that J_φ is positive or negative in B_U. The conclusion follows. $\square$

Proposition 1.14.8. *Let $(\varphi_i : U_i \to V_i)_{i\in I}$ be a family of homeomorphisms such that $X = \bigcup_{i\in I} V_i$ each U_i being an open subset of $\mathbb{R}^n$. Assume that the homeomorphism*

$$\varphi_j^{-1} \circ \varphi_i : \varphi_i^{-1}(V_i \cap V_j) \to \varphi_j^{-1}(V_i \cap V_j)$$

is oriented for any $i, j \in I$. Then, there is a unique orientation of X such that $\varphi_i : U_i \to V_i$ is oriented for any $i \in I$. Moreover, any orientation of X may be obtained in this way.

Proof. The first part is obtained by gluing the isomorphisms

$$\mathbb{Z}_{V_i} \simeq or_{V_i}$$

induced by the isomorphisms $\mathbb{Z}_{U_i} \simeq or_{U_i}$ corresponding to the canonical orientations of the U_i's.

As for the second part, it follows directly from the definition if one keeps in mind that it is always possible to reverse the orientation of a homeomorphism

$$\varphi_i : U_i \to V_i \qquad (U_i \text{ open subset of } \mathbb{R}^n)$$

by composing it with the reflection

$$(x_1, \cdots, x_n) \mapsto (x_1, \cdots, -x_n).$$

□

Corollary 1.14.9. *On a differential manifold, the topological and differential notions of orientation coincide*

Exercise 1.14.10. *Let $\mathcal{F}$ be an abelian sheaf on the topological space X*

(a) Assume U and V are two open subsets of X with $X = U \cup V$ and $U \cap V$ non-empty and connected. Show that if the abelian sheaves $\mathcal{F}_{|U}$ and $\mathcal{F}_{|V}$ are constant, then so is $\mathcal{F}$.

(b) Deduce from (a) that a locally constant sheaf on $[0,1]^n$ is constant.

Remark 1.14.11 (Classification of locally constant sheaves).
Let X be a topological space which is path-connected and locally path-connected, let $\mathcal{F}$ be a locally constant sheaf on X and let $\gamma : [0,1] \to X$ be a continuous path between x and y. It follows from the preceding exercise that we have

$$\gamma^{-1}\mathcal{F} \simeq M_{[0,1]}$$

where M is an abelian group. Hence, the canonical morphisms

$$(\gamma^{-1}\mathcal{F})([0,1]) \to \left(\gamma^{-1}\mathcal{F}\right)_0 \qquad \text{and} \qquad (\gamma^{-1}\mathcal{F})([0,1]) \to \left(\gamma^{-1}\mathcal{F}\right)_1$$

are isomorphisms. This gives us a canonical isomorphism

$$\left(\gamma^{-1}\mathcal{F}\right)_0 \to \left(\gamma^{-1}\mathcal{F}\right)_1$$

and consequently a canonical isomorphism

$$m_\gamma : \mathcal{F}_x \to \mathcal{F}_y.$$

which is called the monodromy along γ. A simple computation shows that if γ' is a path from y to z then $m_{\gamma'} \circ m_\gamma = m_{\gamma' \circ \gamma}$. If the path γ_0 and γ_1 are connected by a homotopy $h : [0,1]^2 \to X$ i.e. if

$$h(t,0) = \gamma_0(t) \qquad \text{and} \qquad h(t,1) = \gamma_1(t);$$

then $h^{-1}\mathcal{F}$ is constant on $[0,1]^2$ and one sees easily that the morphisms

$$m_{\gamma_1} : \mathcal{F}_x \to \mathcal{F}_y \qquad \text{and} \qquad m_{\gamma_2} : \mathcal{F}_x \to \mathcal{F}_y$$

are equal. In particular, $\gamma \mapsto m_\gamma$ induces a representation of the Poincaré group $\pi_1(X,x)$ of X at x on $\mathcal{F}_x$ which is called the monodromy representation of $\mathcal{F}$ at x.

It can be shown that the functor from the category of locally constant sheaves on X to the category of representations of $\pi_1(X,x)$ on abelian groups obtained by associating to a locally constant sheaf $\mathcal{F}$ on X its monodromy representation at x is an equivalence of categories.

A trivial consequence is that locally constant sheaves on X are constant if X is simply path-connected.

Another consequence is that locally constant sheaves with fiber $\mathbb{Z}$ on X correspond to representations of $\pi_1(X,x)$ on $\mathbb{Z}$. Since the only automorphisms of $\mathbb{Z}$ are $\pm\,\mathrm{id}$, these representations may be classified by morphisms

$$\pi_1(X,x) \to \mathbb{Z}_2;$$

non trivial representations corresponding to epimorphisms. Therefore, non constant locally constant sheaves with fiber $\mathbb{Z}$ on X are classified by invariant subgroup of index 2 of $\pi_1(X,x)$.

In particular, if X is a topological manifold and $\pi_1(X,x)$ has no invariant subgroup of index 2, then X is orientable.

Proposition 1.14.12. *Let X be an oriented topological manifold of dimension n. Denote μ_X its orientation class. Then, μ_X may be viewed as an element of $\mathrm{H}_n(X;\mathbb{Z})$ and*

$$\cdot \frown \mu_X : \mathrm{H}^k_\Phi(X;\mathbb{Z}) \to \mathrm{H}^\Phi_{n-k}(X;\mathbb{Z})$$

is an isomorphism.

Proof. Since $\omega_X = or_X[n]$, we have

$$\Gamma(X; or_X) = \mathrm{H}^{-n}(\mathrm{R}\Gamma(X;\omega_X)) = \mathrm{H}_n(X;\mathbb{Z}_X)$$

and μ_X may be viewed as an element of $\mathrm{H}_n(X;\mathbb{Z}_X)$. By the functoriality of the cup product, we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}^k_\Phi(X;\mathbb{Z}_X)\otimes \mathrm{H}^{-n}(X;\omega_X) \arrow[r, "\smile"] & \mathrm{H}^{k-n}_\Phi(X;\omega_X) \\
\mathrm{H}^k_\Phi(X;\mathbb{Z}_X)\otimes \mathrm{H}^{-n}(X;\mathbb{Z}_X[n]) \arrow[r] \arrow[u, "\mathrm{id}\otimes \mathrm{H}^{-n}(X;\nu)"] & \mathrm{H}^{k-n}_\Phi(X;\mathbb{Z}_X[n]) \arrow[u, "\mathrm{H}^{k-n}_\Phi(X;\nu)"']
\end{tikzcd}$$

where $\nu:\mathbb{Z}_X[n]\to\omega_X$ is the isomorphism associated to μ_X. Since μ_X is the image of $1_X\in\Gamma(X;\mathbb{Z}_X)=\mathrm{H}^{-n}(X;\mathbb{Z}_X[n])$ by $\mathrm{H}^{-n}(X;\nu)$, the diagram

$$\begin{tikzcd}
\mathrm{H}^k_\Phi(X;\mathbb{Z}_X) \arrow[r, "\cdot\smile\mu_X"] & \mathrm{H}^{k-n}_\Phi(X;\omega_X) \\
\mathrm{H}^k_\Phi(X;\mathbb{Z}_X) \arrow[r, "\cdot\smile 1_X"'] \arrow[u, "\mathrm{id}"] & \mathrm{H}^{k-n}_\Phi(X;\mathbb{Z}_X[n]) \arrow[u, "\mathrm{H}^{k-n}_\Phi(X;\nu)"']
\end{tikzcd}$$

is commutative and the conclusion follows. ☐

Remark 1.14.13. When we work with oriented topological manifolds, the preceding proposition allows us to transform operations on cohomology into operations on homology and vice-versa. In particular, if $f:X\to Y$ is a continuous map between oriented topological manifolds of dimension n_X, n_Y, then

$$f_*:\mathrm{H}^c_k(X;\mathbb{Z})\to\mathrm{H}^c_k(Y;\mathbb{Z})$$

and

$$f^*:\mathrm{H}^k(Y;\mathbb{Z})\to\mathrm{H}^k(X;\mathbb{Z})$$

induce canonical morphisms

$$f_!:\mathrm{H}^{n_X-k}_c(X;\mathbb{Z})\to\mathrm{H}^{n_Y-k}_c(Y;\mathbb{Z})$$

and

$$f^!:\mathrm{H}_{n_Y-k}(Y;\mathbb{Z})\to\mathrm{H}_{n_X-k}(X;\mathbb{Z}).$$

Moreover,

$$\smile:\mathrm{H}^k_\Phi(X;\mathbb{Z})\otimes\mathrm{H}^l_\Psi(X;\mathbb{Z})\to\mathrm{H}^{k+l}_{\Phi\cap\Psi}(X;\mathbb{Z})$$

gives rise to the intersection product

$$\cdot:\mathrm{H}^\Phi_{n_X-k}(X;\mathbb{Z})\otimes\mathrm{H}^\Psi_{n_X-l}(X;\mathbb{Z})\to\mathrm{H}^{\Phi\cap\Psi}_{n_X-k-l}(X;\mathbb{Z}).$$

Among the many compatibility formulas relating these operations, let us just recall that

$$f_!(f^*c^p\smile c^q)=c^p\smile f_!c^q$$

or dually that

$$f_*(f^! c_p \cdot c_q) = c_p \cdot f_*(c_q)$$

and that

$$(c_p \times c_q) \cdot (c_r \times c_s) = (-1)^{(n_X - p)(n_Y - s)}(c_p \cdot c_r) \times (c_q \cdot c_s).$$

Definition 1.14.14. Let X, Y be oriented topological manifolds of dimension n and p. Assume Y is a closed subspace of X and denote $i : Y \to X$ the inclusion map. Then, the *fundamental homology class of Y in X* is the class

$$[Y] = i_*(\mu_Y)$$

of $\mathrm{H}_p^Y(X;\mathbb{Z})$. Dually, the *fundamental cohomology class (or Thom class) of Y in X* is the class

$$\tau_{Y/X} = i_!(1_Y)$$

of $\mathrm{H}_Y^{n-p}(X;\mathbb{Z})$ which is also characterized by the formula

$$\tau_{Y/X} \frown \mu_X = [Y].$$

Proposition 1.14.15. *Let X be an oriented differential manifold of dimension n and let Y, Z be closed oriented differential submanifolds of dimension p and q of X. Assume Y and Z meet transversally. Then $Y \cap Z$ is a closed differential submanifold of dimension $p + q - n$ of X which is canonically oriented. Moreover, for this orientation, we have*

$$[Y \cap Z] = [Y] \cdot [Z]$$

and

$$\tau_{Y \cap Z/X} = \tau_{Y/X} \smile \tau_{Z/X}.$$

Proof. We will only treat the case where $n > p$, $n > q$, $p + q > n$, leaving the adaptation to the other cases to the reader. Since Y and Z meet transversally, we have

$$T_xX = T_xY + T_xZ \quad \text{and} \quad T_x(Y \cap Z) = T_xY \cap T_xZ$$

at any point $x \in Y \cap Z$. It follows that it is possible to find an oriented basis of T_xX of the form

$$u_1, \ldots, u_{n-q}, w_1, \ldots, w_{p+q-n}, v_1, \ldots, v_{n-p}$$

where

$$u_1, \ldots, u_{n-q}, w_1, \ldots, w_{p+q-n} \quad \text{and} \quad w_1, \ldots, w_{p+q-n}, v_1, \ldots, v_{n-p}$$

are oriented bases of T_xY and T_xZ and

$$w_1, \dots, w_{p+q-n}$$

is a basis of $T_x(Y \cap Z)$. We define the canonical orientation of $Y \cap Z$ as the one corresponding to such a basis.

Let us now prove that

$$[Y \cap Z] = [Y] \cdot [Z]$$

in $\mathrm{H}^Y_{p+q-n}(X;\mathbb{Z})$. Since

$$U \mapsto \mathrm{H}^{U\cap Y}_{p}(U;\mathbb{Z}), \quad U \mapsto \mathrm{H}^{U\cap Z}_{q}(U;\mathbb{Z}), \quad U \mapsto \mathrm{H}^{U\cap Y\cap Z}_{p+q-n}(U;\mathbb{Z})$$

are sheaves on X, the problem is of local nature. Therefore, we may assume that

$$X = \mathbb{R}^{n-q} \times \mathbb{R}^{p+q-n} \times \mathbb{R}^{n-p}$$

and that

$$Y = \mathbb{R}^{n-q} \times \mathbb{R}^{p+q-n} \times \{0\}, \quad Z = \{0\} \times \mathbb{R}^{p+q-n} \times \mathbb{R}^{n-p}$$

the orientations being the products of the canonical orientations of the factors. In this case,

$$Y \cap Z = \{0\} \times \mathbb{R}^{p+q-n} \times \{0\}$$

with the orientation given by the canonical orientation of $\mathbb{R}^{p+q-n}$. Therefore,

$$[Y] = \mu_{\mathbb{R}^{n-q}} \times \mu_{\mathbb{R}^{p+q-n}} \times [0], \quad [Z] = [0] \times \mu_{\mathbb{R}^{p+q-n}} \times \mu_{\mathbb{R}^{n-p}}$$

and

$$[Y \cap Z] = [0] \times \mu_{\mathbb{R}^{p+q-n}} \times [0].$$

Since by Remark 1.14.13 we have

$$\begin{aligned}
&(\mu_{\mathbb{R}^{n-q}} \times \mu_{\mathbb{R}^{p+q-n}} \times [0]) \cdot ([0] \times \mu_{\mathbb{R}^{p+q-n}} \times \mu_{\mathbb{R}^{n-p}}) \\
&\quad = (\mu_{\mathbb{R}^{n-q}} \cdot [0]) \times (\mu_{\mathbb{R}^{p+q-n}} \cdot \mu_{\mathbb{R}^{p+q-n}}) \times ([0] \cdot \mu_{\mathbb{R}^{n-p}}) \\
&\quad = [0] \times \mu_{\mathbb{R}^{p+q-n}} \times [0]
\end{aligned}$$

the conclusion follows. $\square$

1.15 Sheaves of rings and modules

In order to focus the survey contained in the preceding sections on the basic ideas of sheaf theory, we have chosen to deal only with sheaves of abelian groups. In the rest of this book, we will however need to use sheaves of rings and modules. The adaptation of the theory reviewed above to this more general situation being rather mechanical, we will not do it explicitly here and refer the interested reader to standard texts on the subject.

2

Euler class of manifolds and real vector bundles

2.1 Lefschetz fixed point formula

Let X, Y be two compact oriented topological manifolds of dimension n.

Recall that a correspondence between X and Y is a subset C of $X \times Y$. The image of a subset $A \subset X$ by the correspondence C is the set

$$C(A) = \{y \in Y : \exists x \in A, (x, y) \in C\}$$

which may also be described as

$$p_Y(p_X^{-1}(A) \cap C).$$

Following Lefschetz, we will introduce similar notions in the framework of homology.

Definition 2.1.1. A *homological correspondence* from X to Y is a class $\gamma_n \in \mathrm{H}_n(X \times Y; \mathbb{Z})$.

For any abelian group M, the *image* of a class $c_p \in \mathrm{H}_p(X; M)$ by γ_n is the class

$$\gamma_n(c_p) = (p_Y)_*(p_X^!(c_p) \cdot \gamma_n) \in \mathrm{H}_p(Y; M).$$

Let $f : X \to Y$ be a continuous map. As usual, set $\delta_f(x) = (x, f(x))$ and let

$$\Delta_f = \mathrm{Im}\, \delta_f = \{(x, y) : y = f(x)\}$$

denote the graph of f. Clearly,

$$\delta_f : X \to X \times Y$$

induces a homeomorphism between X and Δ_f which turns Δ_f into a compact oriented manifold of dimension n. We will denote

$$i_f : \Delta_f \to X \times Y$$

the canonical inclusion. As is well-known, we may recover f from the correspondence Δ_f. A similar result is true in homology.

Proposition 2.1.2. *The class*

$$\gamma_f = (\delta_f)_*(\mu_X) = (i_f)_*(\mu_{\Delta_f})$$

is a homological correspondence from X to Y for which we have

$$\gamma_f(c_p) = f_*(c_p).$$

Proof. Recall that

$$[\Delta_f] = (i_f)_*(\mu_{\Delta_f}) = (\delta_f)_*(\mu_X).$$

Recall also that $c_p \cdot \mu_X = c_p$. As a matter of fact, for $c_p = c^{n-p} \frown \mu_X$, we have

$$c_p \cdot \mu_X = (c^{n-p} \frown \mu_X) \cdot \mu_X = (c^{n-p} \smile 1) \frown \mu_X = c_p.$$

Therefore, keeping in mind that

$$p_X \circ \delta_f = \mathrm{id}_X, \qquad p_Y \circ \delta_f = f,$$

we have successively

$$\begin{aligned} \gamma_f(c_p) &= (p_Y)_*(p_X^! c_p \cdot (\delta_f)_*(\mu_X)) \\ &= (p_Y)_*(\delta_f)_*(\delta_f^! p_X^! c_p \cdot \mu_X) \\ &= f_*(c_p \cdot \mu_X) \end{aligned}$$

□

For $p \in \mathbb{N}$, let $\alpha_{p,r}$ $(r = 1, \cdots, R_p)$ be a basis of the finite dimension vector space $\mathrm{H}_p(X;\mathbb{Q})$. Thanks to Poincaré-Verdier duality, we know that the pairing

$$\mathrm{H}_p(X;\mathbb{Q}) \times \mathrm{H}_{n-p}(X;\mathbb{Q}) \to \mathbb{Q}$$

which sends (c_p, c_{n-p}) to $\#(c_p \cdot c_{n-p})$ is perfect. Therefore, there is a unique basis $\alpha^\vee_{p,r}$ $(r = 1, \cdots, R_p)$ of $\mathrm{H}_p(X; \mathbb{Q})$ such that

$$\#(\alpha^\vee_{p,r} \cdot \alpha_{n-p,r'}) = \delta_{rr'}$$

for any r, $r' \in \{1, \cdots, R_p\}$. Denote $\beta_{q,s}$, $\beta^\vee_{q,s}$ $(s = 1, \cdots, S_q)$ similar basis for the rational homology of Y.

Proposition 2.1.3. *Let γ be a homological correspondence from X to Y. Then,*

$$\gamma = \sum_{p=1}^{n} \sum_{r=1}^{R_p} \sum_{s=1}^{S_p} (\Gamma_p)_{sr} \alpha^\vee_{n-p,r} \times \beta_{p,s}$$

where Γ_p is the matrix of

$$\gamma(\cdot) : \mathrm{H}_p(X; \mathbb{Q}) \to \mathrm{H}_p(Y; \mathbb{Q})$$

with respect to the basis $\alpha_{p,r}$ $(r = 1, \cdots, R_p)$ and $\beta_{p,s}$ $(s = 1, \cdots, S_p)$.

Proof. Thanks to Künneth theorem, we know that the classes

$$\alpha^\vee_{p,r} \times \beta_{n-p,s} \qquad (p = 1, \cdots, n;\ r = 1, \cdots, R_p;\ s = 1, \cdots, S_p)$$

form a basis of $\mathrm{H}_n(X \times Y; \mathbb{Q})$. Hence

$$\gamma = \sum_{p=1}^{n} \sum_{r=1}^{R_p} \sum_{s=1}^{S_p} (\Gamma_p)_{sr} \alpha^\vee_{n-p,r} \times \beta_{p,s}$$

where the $(\Gamma_p)_{sr}$ are rational numbers. Let us compute $\gamma(\alpha_{p_0,r_0})$. Note that

$$p^!_X(c_p) = (-1)^{n(n-p)} c_p \times \mu_Y.$$

As a matter of fact, we may assume $c_p = c^{n-p} \frown \mu_X$. Hence,

$$\begin{aligned} p^!_X(c_p) &= p^*_X(c^{n-p}) \frown \mu_{X \times Y} \\ &= (c^{n-p} \times 1_Y) \frown (\mu_X \times \mu_Y) \\ &= (-1)^{n(n-p)} (c^{n-p} \frown \mu_X) \times (1_Y \frown \mu_Y) && (1) \\ &= c_p \times \mu_Y && (2) \end{aligned}$$

where in (1) we have used the formula

$$(c^p \times c^q) \frown (c_r \times c_s) = (-1)^{p(s-q)} (c^p \frown c_r) \times (c^q \frown c_s)$$

and in (2) the formula

$$1_Y \frown c_p = c_p.$$

Therefore, $\gamma(\alpha_{p_0,r_0})$ is equal to

$$\sum_{p=1}^{n}\sum_{r=1}^{R_p}\sum_{s=1}^{S_p}(-1)^{n(n-p_0)}(\Gamma_p)_{sr}(p_Y)_*\left[(\alpha_{p_0,r_0}\times\mu_Y)\cdot(\alpha^\vee_{n-p,r}\times\beta_{p,s})\right].$$

Using the formula

$$(\alpha_p\times\alpha_q)\cdot(\beta_r\times\beta_s)=(-1)^{(n-p)(n-s)}(\alpha_p\cdot\beta_r)\times(\alpha_q\cdot\beta_s)$$

we see that

$$\begin{aligned}
&(p_Y)_*\left[(\alpha_{p_0,r_0}\times\mu_Y)\cdot(\alpha^\vee_{n-p,r}\times\beta_{p,s})\right]\\
&\quad=(-1)^{(n-p_0)(n-p)}(p_Y)_*\left[\alpha_{p_0,r_0}\cdot(\alpha^\vee_{n-p,r})\times(\mu_Y\cdot\beta_{p,s})\right]\\
&\quad=(-1)^{(n-p_0)(n-p)}\#(\alpha_{p_0,r_0}\cdot\alpha^\vee_{n-p,r})\beta_{p,s} &\text{(*)}\\
&\quad=(-1)^{(n-p_0)(n-p)}(-1)^{p(n-p_0)}\delta_{p,p_0}\delta_{r,r_0}\beta_{p,s}
\end{aligned}$$

where (*) follows from the formula $(f\times g)_*(\alpha_p\times\alpha_q)=f_*(\alpha_p)\times g_*(\alpha_q)$. Therefore,

$$\begin{aligned}
\gamma(\alpha_{p_0,r_0})&=\sum_{p=1}^{n}\sum_{r=1}^{R_p}\sum_{s=1}^{S_p}(\Gamma_p)_{sr}\delta_{p,p_0}\delta_{r,r_0}\beta_{p,s}\\
&=\sum_{r=1}^{R_p}\sum_{s=1}^{S_p}(\Gamma_{p_0})_{sr_0}\beta_{p_0,s}
\end{aligned}$$

and the conclusion follows. ☐

Recall that a fixed point of a correspondence C from X to X is a point $x\in X$ such that $x\in C(x)$. Such a point is characterized by the fact that $(x,x)\in C\cap\Delta$ where Δ is the diagonal of $X\times X$.

Definition 2.1.4. Let $\gamma\in\mathrm{H}_n(X\times X;\mathbb{Z})$ be a homological correspondence from X to X. We define the *algebraic number of fixed points of* γ as the number

$$\#(\gamma\cdot[\Delta])$$

and the *Lefschetz number of* γ as the number

$$\Lambda_\gamma=\sum_{p=0}^{n}(-1)^p\operatorname{tr}\left[\gamma(\cdot):\mathrm{H}_p(X;\mathbb{Q})\to\mathrm{H}_p(X;\mathbb{Q})\right].$$

Theorem 2.1.5 (Lefschetz fixed points formula). *Assume*

$$\gamma\in\mathrm{H}_n(X\times X;\mathbb{Z})$$

is a homological correspondence from X to X. Then,

$$\#(\gamma \cdot [\Delta]) = \Lambda_\gamma.$$

Proof. Using the classes $\alpha_{p,r}$, $\alpha^\vee_{p,r}$ introduced above and Proposition 2.1.3, we see that

$$\gamma = \sum_{p=1}^{n} \sum_{r=1}^{R_p} \sum_{s=1}^{R_p} (\Gamma_p)_{sr} \alpha^\vee_{n-p,r} \times \alpha_{p,s}$$

where Γ_p is the matrix of

$$\gamma(\cdot) : \mathrm{H}_p(X; \mathbb{Q}) \to \mathrm{H}_p(X; \mathbb{Q})$$

with respect to the basis $\alpha_{p,r}$ $(r = 1, \cdots, R_p)$. Similarly, we have

$$\gamma_{\mathrm{id}} = \sum_{p'=1}^{n} \sum_{r'=1}^{R_p} \sum_{s'=1}^{R_p} \delta_{s',r'} \alpha^\vee_{n-p',r'} \times \alpha_{p',s'}.$$

Let $\sim : X \times X \to X \times X$ be the morphism defined by setting

$$\sim (x, x') = (x', x).$$

We have

$$\sim_* (\gamma_{\mathrm{id}}) = \sum_{p'=1}^{n} \sum_{r'=1}^{R_p} \sum_{s'=1}^{R_p} (-1)^{p'(n-p')} \delta_{s',r'} \alpha_{p',s'} \times \alpha^\vee_{n-p',r'}$$

and

$$\sim_* (\gamma_{\mathrm{id}}) = \sim_* \delta_* \mu_X = \delta_* \mu_X = \gamma_{\mathrm{id}}.$$

Therefore,

$$[\Delta] = \sum_{p'=1}^{n} \sum_{r'=1}^{R_p} \sum_{s'=1}^{R_p} (-1)^{p'(n-p')} \delta_{s',r'} \alpha_{p',s'} \times \alpha^\vee_{n-p',r'}.$$

Since

$$\# \left((\alpha^\vee_{n-p,r} \times \alpha_{p,s}) \cdot (\alpha_{p',s'} \times \alpha^\vee_{n-p',r'}) \right) = (-1)^{pp'} (-1)^{p'(n-p'} \delta_{pp'} \delta_{rs'} \delta_{sr'}$$

we see that

$$\#(\gamma \cdot [\Delta]) = \sum_{p=1}^{n} \sum_{r=1}^{R_p} \sum_{s=1}^{R_p} (-1)^{p^2} (\Gamma_p)_{sr} \delta_{sr} = \sum_{p=1}^{n} (-1)^{p^2} \operatorname{tr} \Gamma_p$$

and the conclusion follows since $p^2 \equiv p \pmod 2$. $\square$

Corollary 2.1.6.

(a) Let $f : X \to X$ be a continuous map. Set $\Lambda_f = \Lambda_{\gamma_f}$. Then,

$$\#(\gamma_f \cdot [\Delta]) = \Lambda_f = \sum_{p=0}^{n} (-1)^p \operatorname{tr}(f_* : \mathrm{H}_p(X;\mathbb{Q}) \to \mathrm{H}_p(X;\mathbb{Q})).$$

In particular, $\Lambda_f = 0$ if f has no fixed point.

(b) For $f = \mathrm{id}_X$, we get

$$\#([\Delta] \cdot [\Delta]) = \Lambda_{\mathrm{id}} = \sum_{p=0}^{n} (-1)^p \dim_{\mathbb{Q}} \mathrm{H}_p(X;\mathbb{Q}) = \chi(X).$$

Exercise 2.1.7. *Assume $n \in \mathbb{N}_0$. Let $f : S_n \to S_n$ be a continuous map. Define the degree of f as the unique integer* $\deg(f)$ *making the diagram*

$$\begin{tikzcd} \mathrm{H}^n(S_n;\mathbb{Z}) \arrow[r, "f^*"] & \mathrm{H}^n(S_n;\mathbb{Z}) \\ \mathbb{Z} \arrow[u, "\cdot v_{S_n}", "\wr"'] \arrow[r, "\cdot \deg(f)"'] & \mathbb{Z} \arrow[u, "\cdot v_{S_n}"', "\wr"] \end{tikzcd}$$

commutative.

(a) Show that

$$\Lambda_f = 1 + (-1)^n \deg(f).$$

(b) Deduce from (a) that f has a fixed point if $\deg(f) \neq (-1)^{n+1}$.

(c) Apply (b) to show that if there is a homotopy connecting f to id_{S_n} *and n is even then f has a fixed point.*

(d) As an application, show that any vector field θ of class C_1 on S_n must vanish at some $x \in S_n$ if n is even.

Solution. (a) follows directly from the definition of $\deg(f)$ and the fact that

$$\mathrm{H}^k(S_n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0, n; \\ 0 & \text{otherwise.} \end{cases}$$

(b) Since $\Lambda_f = 0$ if and only if $\deg(f) = (-1)^{n+1}$, the conclusion follows directly from Corollary 2.1.6.

(c) When $f \sim \mathrm{id}_{S_n}$, we have $f^* = (\mathrm{id}_{S_n})^* = \mathrm{id}$. Hence, $\deg(f) = 1$ and the conclusion follows from (b).

(d) Let $\varphi_t(x)$ be the flow of the vector field θ. From (c), we know that φ_t has a fixed point $x_t \in S_n$ for any $t \in \mathbb{R}$. Since S_n is compact, we may find a sequence $t_k \to 0^+$ such that $x_{t_k} \to x_0 \in S_n$. Let $t \in \mathbb{R}$ and choose the sequence m_k of integers such that $m_k t_k \to t$. Clearly,

$$\varphi_t(x_0) = \lim_{k \to \infty} \varphi_{m_k t_k}(x_{t_k}) = \lim_{k \to \infty} x_{t_k} = x_0.$$

It follows that $\varphi_t(x_0) = x_0$ for any $t \in \mathbb{R}$ and hence that $\theta(x_0) = 0$. □

Exercise 2.1.8.

(a) Let X be a compact oriented manifold of dimension n. Show that

$$\chi(X) = 0$$

if n is odd.

(b) Let X be a compact oriented C_1 manifold. Show that if X has a nowhere vanishing C_1 vector field then $\chi(X) = 0$.

Solution. (a) Set

$$R_p = \dim \mathrm{H}_p(X; \mathbb{Q}).$$

By Poincaré duality, we have $R_p = R_{n-p}$. Therefore, if $n = 2k+1$ ($k \in \mathbb{N}$) we have

$$\begin{aligned}
\chi(X) = \sum_{p=0}^{n} (-1)^p R_p &= \sum_{p=0}^{k} (-1)^p R_p + \sum_{p=k+1}^{2k+1} (-1)^p R_p \\
&= \sum_{p=0}^{k} (-1)^p R_p + \sum_{p=0}^{k} (-1)^{n-p} R_{n-p} \\
&= 0.
\end{aligned}$$

(b) Working as in Exercise 2.1.7, we see that $\chi(X) \neq 0$ entails that any C_1-vector field on X vanish for some $x \in X$. The conclusion follows. □

Corollary 2.1.9. *Let X be a compact differential manifold of dimension n.*

(a) Assume C is a closed differential submanifold of dimension n of $X \times X$ which meets Δ transversally. Then $C \cap \Delta$ is finite and

$$\Lambda_{[C]} = \sum_{(x,x) \in C \cap \Delta} i_{(x,x)}(C, \Delta)$$

where $i_{(x,x)}(C, \Delta)$ is equal to 1 if the canonical orientation of

$$T_{(x,x)}C \oplus T_{(x,x)}\Delta$$

coincides with that of $T_{(x,x)}(X \times X)$ and to -1 otherwise.

(b) *Assume $f : X \to X$ is a differentiable map with a non-empty set F of fixed points. Assume that for any $x \in F$,*

$$f'_x : T_xX \to T_xX$$

does not fix any non-zero tangent vector. Then, the set F is finite and

$$\Lambda_f = \sum_{x\in F} \operatorname{sgn}(\det(\operatorname{id} - f'_x)).$$

Proof. (a) Thanks to Theorem 2.1.5, the result follows directly from Proposition 1.14.15.

(b) Clearly, $(x,x) \in \Delta_f \cap \Delta$ if and only if $x \in F$. Moreover, for any $x \in F$, we have

$$T_{(x,x)}\Delta_f = \{(\theta, f'_x(\theta)) : \theta \in T_xX\}$$

and

$$T_{(x,x)}\Delta = \{(\theta, \theta) : \theta \in T_xX\}.$$

Hence, we have

$$T_{(x,x)}\Delta_f \cap T_{(x,x)}\Delta = \{(\theta, \theta) : \theta \in T_xX, f'_x(\theta) = 0\}.$$

It follows that Δ_f and Δ meet transversally. If $\theta_1, \dots, \theta_n$ is an oriented basis of T_xX, then

$$(\theta_1, f'_x(\theta_1)), \dots, (\theta_n, f'_x(\theta_n))$$

and

$$(\theta_1, \theta_1), \dots, (\theta_n, \theta_n)$$

are oriented bases for $T_{(x,x)}\Delta_f$ and $T_{(x,x)}\Delta$. Therefore,

$$(\theta_1, f'_x(\theta_1)), \dots, (\theta_n, f'_x(\theta_n)), (\theta_1, \theta_1), \dots, (\theta_n, \theta_n)$$

is an oriented basis for $T_{(x,x)}\Delta_f \oplus T_{(x,x)}\Delta$. This basis has the same orientation as the basis

$$(0, f'_x(\theta_1) - \theta_1), \dots, (0, f'_x(\theta_n) - \theta_n), (\theta_1, \theta_1), \dots, (\theta_n, \theta_n).$$

Using the assumptions, we know that $f'_x - \operatorname{id}$ is injective and hence bijective. It follows that

$$f'_x(\theta_1) - \theta_1, \dots, f'_x(\theta_n) - \theta_n$$

is a basis of T_xX. Hence,

$$(0, f'_x(\theta_1) - \theta_1), \dots, (0, f'_x(\theta_n) - \theta_n), (\theta_1, 0), \dots, (\theta_n, 0).$$

is an oriented basis for $T_{(x,x)}\Delta_f \oplus T_{(x,x)}\Delta$ and

$$i_{(x,x)}(\Delta_f, \Delta) = (-1)^n \operatorname{sgn}(\det(f'_x - \operatorname{id})) = \operatorname{sgn}(\det(\operatorname{id} - f'_x)).$$

The conclusion follows from (a). ☐

2.2 Euler classes of manifolds and index theorem

Definition 2.2.1. The *Euler class* of an oriented topological manifold X is the class

$$\epsilon_X = \delta^* \tau_{\Delta/X\times X}$$

Proposition 2.2.2. *For any oriented compact topological manifold, we have*

$$\chi(X) = \int_X \epsilon_X.$$

Proof. We know that

$$\chi(X) = \#([\Delta]\cdot[\Delta]).$$

Therefore,

$$\begin{aligned}
\chi(X) &= \#((\tau_{\Delta/X\times X} \smile \tau_{\Delta/X\times X}) \frown \mu_{X\times X}) \\
&= \int_{X\times X} \tau_{\Delta/X\times X} \smile \tau_{\Delta/X\times X} = \int_{X\times X} \tau_{\Delta/X\times X} \smile \delta_!(1_X) \\
&= \int_{X\times X} \delta_!(\delta^* \tau_{\Delta/X\times X}) = \int_X \epsilon_X
\end{aligned}$$

□

2.3 Basic notions on real vector bundles

A *(continuous) real vector bundle of rank* r is the data of a continuous map

$$p_E : E \to B_E$$

between topological spaces together with structures of real vector spaces of $\dim r$ on each fiber $E_b = p_E^{-1}(b)$ $(b \in B)$ of p. These data being such that for any $b \in B$ there is a neighborhood U of b in B and a family $(e_1, \cdots, e_r)$ of continuous sections of $p_{E|p_E^{-1}(U)} : p_E^{-1}(U) \to U$ with the property that $(e_1(b'), \cdots, e_r(b'))$ is a basis of $p_E^{-1}(b')$ for any $b' \in U$. We call p_E (resp. E, B_E) the *projection* (resp. the *total space*, the *base*) of the vector bundle. A family like $(e_1, \cdots, e_r)$ is called a (continuous) *frame* of the real vector bundle over U. We will often refer to a vector bundle by giving its total space alone assuming that the projection and the basis are clear from the context.

Let E be a real vector bundle of rank r and let U be an open subset of its base B_E. Then, $p_{E|p_E^{-1}(U)} : p_E^{-1}(U) \to U$ has clearly a canonical structure of real vector bundle of rank r. We call it the *restriction of E to U* and denote it by $E_{|U}$.

A *morphism* between two real vector bundles E, F with common base B is a continuous map $f : E \to F$ such that $p_F \circ f = p_E$, the map $f_b : E_b \to F_b$ induced by f being $\mathbb{R}$-linear for any $b \in B$. Clearly, morphisms of real vector bundles may be composed and $\mathbb{R}$-linearly combined in a natural way. Hence, real vector bundles with base B form a category. We will denote it by $\mathcal{V}\mathrm{ect}_{\mathbb{R}}(B)$. One sees easily that this category is additive, the *direct sum* $E \oplus F$ of two real vector bundles E and F with base B being characterized by the fact that

$$(E \oplus F)_b = E_b \oplus F_b,$$

a local frame of $E \oplus F$ being given by $(e_1, \cdots, e_r, f_1, \cdots, f_s)$ if $(e_1, \cdots, e_r)$ and $(f_1, \cdots, f_s)$ are local frames of E and F.

A *trivial* vector bundle is a vector bundle of the form

$$p_B : V \times B \to B$$

where V is a real vector space. A vector bundle is *trivializable* if it is isomorphic to a trivial vector bundle. By definition, any vector bundle is locally of this type.

A *sub-bundle* of a vector bundle F with basis B is the data of a vector bundle E with basis B together with a morphism $i : E \to F$ for which $i_b : E_b \to F_b$ is injective for any $b \in B$.

A *quotient bundle* of a vector bundle E with basis B is the data of a vector bundle F with basis B together with a morphism $q : E \to F$ for which $q_b : E_b \to F_b$ is surjective for any $b \in B$.

Let $i : E \to F$ be a sub-bundle of F. Then, i has a cokernel in $\mathcal{V}\mathrm{ect}_{\mathbb{R}}(B)$ characterized by the fact that

$$(\operatorname{Coker} i)_b = \operatorname{Coker} i_b,$$

a local frame of $\operatorname{Coker} i$ being obtained by considering a local frame of F of the type $(i(e_1), \cdots, i(e_r), f_{r+1}, \cdots, f_s)$ with $(e_1, \cdots, e_r)$ a local frame of E and taking the images of $f_{r+1}, \cdots, f_s$ in $\operatorname{Coker} i$. From this construction, it follows that $\operatorname{Coker} i$ together with the canonical morphism $F \to \operatorname{Coker} i$ is a quotient bundle of F that one often denotes F/E when i is clear from the context.

Let $q : E \to F$ be a quotient bundle of E. Then, q has a *kernel* in $\mathcal{V}\mathrm{ect}_{\mathbb{R}}(B)$ characterized by the fact that

$$(\operatorname{Ker} q)_b = \operatorname{Ker} q_b,$$

a local frame of $\operatorname{Ker} i$ being obtained by considering a local frame of E of the type $(f_1, \cdots, f_r, f_{r+1}, \cdots, f_s)$ where $(q(f_{r+1}), \cdots, q(f_s))$ is a local frame of

F and viewing $(f_1, \cdots, f_r)$ as local sections of $\operatorname{Ker} q$. Clearly, $\operatorname{Ker} i$ together with the canonical morphism $j : \operatorname{Ker} i \to E$ is a sub-bundle of E.

Note that if $i : E \to F$ (resp. $q : E \to F$) is a sub-bundle of F (resp. a quotient bundle of E) then

$$E \simeq \operatorname{Ker}(F \to \operatorname{Coker} i) \qquad (\text{resp. } F \simeq \operatorname{Coker}(\operatorname{Ker} q \to E)).$$

Note also that when the base space B is paracompact, one can show, using a partition of unity, that for any sub-vector bundle E of F one has $F \simeq E \oplus F/E$.

Finally, let us recall that the *inverse image* of vector bundle E of base B by a continuous map $f : B' \to B$ is the vector bundle $f^{-1}(E)$ with base B' characterized by the fact that

$$f^{-1}(E)_{b'} = E_{f(b')}$$

for any $b' \in B'$, a frame of $f^{-1}(E)$ on $f^{-1}(U)$ being given by $(e_1 \circ f, \cdots, e_r \circ f)$ where $(e_1, \cdots, e_r)$ is a frame of E on U.

2.4 Orientation of real vector bundles

Let A be a noetherian ring with finite global homological dimension and let E be a real vector bundle of rank r and base B.

Definition 2.4.1. The *relative dualizing complex of B in E for sheaves of A-modules* is the complex

$$\omega^A_{B/E} = (\mathrm{R}\Gamma_B A_E)_{|B}.$$

Proposition 2.4.2.

(a) *The canonical restriction morphism*

$$Rp(\mathrm{R}\Gamma_B A_E) \to (\mathrm{R}\Gamma_B A_E)_{|B}$$

is an isomorphism. In particular,

$$\mathrm{R}\Gamma(U; \omega^A_{B/E}) \simeq \mathrm{R}\Gamma_U(p^{-1}(U); A)$$

for any open subspace U of X.

(b) *The canonical restriction morphism*

$$(\omega^A_{B/E})_b \to \mathrm{R}\Gamma_{\{b\}}(E_b; A)$$

is an isomorphism. In particular, $\omega^A_{B/E}$ is concentrated in degree r.

Proof. (a) One checks easily by working at the level of fibers that

$$p(\Gamma_B\mathcal{F}) \to (\Gamma_B\mathcal{F})_{|B}$$

is an isomorphism for any flabby sheaf $\mathcal{F}$ on E. The first part follows. As for the second part, it is a consequence of Leray theorem.

(b) The problem being local on B, we may assume $E = \mathbb{R}^r \times B$. Consider the morphism of distinguished triangles

$$\begin{array}{ccccccc}
[Rp(\mathrm{R}\Gamma_U A_{\mathbb{R}^r\times U})]_b & \longrightarrow & [Rp(A_{\mathbb{R}^r\times U})]_b & \longrightarrow & [R\dot{p}(A_{\dot{E}})]_b & \xrightarrow{+1} & \\
\downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle (2)} & & \downarrow{\scriptstyle (3)} & & \\
\mathrm{R}\Gamma_{\{b\}}(\mathbb{R}^r \times \{b\}; A) & \longrightarrow & \mathrm{R}\Gamma(\mathbb{R}^r \times \{b\}; A) & \longrightarrow & \mathrm{R}\Gamma(\mathbb{R}^r \setminus \{0\} \times \{b\}; A) & \xrightarrow{+1} &
\end{array}$$

We will prove that (2) and (3) are isomorphisms. This will show that (1) is also an isomorphism and the conclusion will follow.

By simple homotopy arguments, one gets the isomorphisms

$$[Rp(A_{\mathbb{R}^r\times U})]_b \simeq A$$

and

$$\mathrm{R}\Gamma(\mathbb{R}^r \times \{b\}; A) \simeq A.$$

Since these isomorphisms transform (2) into the identity, (2) is also an isomorphism.

Denote $q : S_{r-1} \times U \to U$ the second projection. Working as above one sees that (3) will be an isomorphism if the canonical morphism

$$[Rq(A_{S_{r-1}\times U})]_b \to \mathrm{R}\Gamma(S_{r-1} \times \{b\}; A)$$

is an isomorphism. Since S_{r-1} is compact, the fiber of q at b is compact and relatively Haussdorf in $S_{r-1} \times U$. It is thus a taut subspace and the conclusion follows from the fiber formula for Rq □

Definition 2.4.3. The *relative A-orientation sheaf of B in E* is the sheaf

$$\mathrm{or}^A_{B/E} = H^r\omega^A_{B/E}$$

Proposition 2.4.4.

(a) We have

$$\mathrm{R}\Gamma(U; \mathrm{or}^A_{B/E}) \simeq \mathrm{R}\Gamma_U(p^{-1}(U); A)[r]$$

and in particular

$$\Gamma(U; \mathrm{or}^A_{B/E}) \simeq \mathrm{H}^r_U(p^{-1}(U); A)$$

(b) *The canonical restriction morphism*

$$[\mathrm{or}^A_{B/E}]_b \to \mathrm{H}^r_{\{b\}}(E_b; A)$$

is an isomorphism.

(c) *Any frame* $(e_1, \cdots, e_r)$ *of* E *on* U *induces a canonical isomorphism*

$$\psi_{(e_1,\cdots,e_r)} : A_U \xrightarrow{\sim} (\mathrm{or}^A_{B/E})_{|U}.$$

In particular, $\mathrm{or}^A_{B/E}$ *is a locally constant sheaf with fiber* A.

(d) *Let* $(e_1, \cdots, e_r)$ *and* $(e'_1, \cdots, e'_r)$ *be two frames of* E *on* U. *Set*

$$e'_i(u) = \sum_{s=1}^{r} S_{ji}(u) e_j(u)$$

for $i = 1, \cdots, r$. *Assume* $\det S(u) \gtrless 0$ *on* U. *Then,*

$$\psi_{(e'_1,\cdots,e'_r)} = \pm\psi_{(e_1,\cdots,e_r)}.$$

Proof. (a) It follows from part (b) of the preceding proposition that

$$\omega^A_{B/E} \simeq \mathrm{or}^A_{B/E}[-r].$$

Therefore,

$$\mathrm{R}\Gamma(U; \mathrm{or}^A_{B/E}) \simeq \mathrm{R}\Gamma(U; \omega^A_{B/E})[r]$$

and the conclusion follows from part (a) of preceding proposition.

(b) This follows directly from part (b) of the preceding proposition.

(c) Let $(e_1, \cdots, e_r)$ be a frame of E on U. Consider the morphism

$$\varphi : \mathbb{R}^r \times U \to p^{-1}(U)$$

defined by setting

$$\varphi(x_1, \cdots, x_r, u) = \sum_{j=1}^{r} x_j e_j(u).$$

Clearly, φ is an isomorphism of real vector bundles. It follows that the canonical morphism

$$\varphi^* : \mathrm{or}^A_{U/p^{-1}(U)} \to \mathrm{or}^A_{U/\mathbb{R}^r \times U} \tag{*}$$

is an isomorphism. By pull-back through the first projection, the canonical generator

$$v_{\{0\}} \in \mathrm{H}^r_{\{0\}}(\mathbb{R}^r; A)$$

gives the canonical class

$$p_{\mathbb{R}^r}^* v_{\{0\}} \in \mathrm{H}^r_U(\mathbb{R}^r \times U; A).$$

Thanks to the second isomorphism of (a), we get a canonical class

$$\mu_{U/\mathbb{R}^r \times U} \in \Gamma(U; \mathrm{or}^A_{U/\mathbb{R}^r \times U}).$$

This class induces a morphism

$$A_U \to \mathrm{or}^A_{U/\mathbb{R}^r \times U}\,. \tag{**}$$

which corresponds at the level of fibers to the isomorphism

$$A \to \mathrm{H}^r_{\{0\}}(\mathbb{R}^r; A)$$

induced by $v_{\{0\}}$. It follows that (**) is an isomorphism. By combining it with (*), we get the conclusion.

(c) Consider the isomorphism

$$\sigma : \mathbb{R}^r \times U \to \mathbb{R}^r \times U$$

defined by setting $\sigma(x', u) = (S(u)x', u)$ and assume $\det S(u) > 0$ (resp. $\det S(u) < 0$) on U. To conclude, it is sufficient to prove that the canonical morphism

$$\sigma^* : \mathrm{or}^A_{U/\mathbb{R}^r \times U} \to \mathrm{or}^A_{U/\mathbb{R}^r \times U}$$

sends $\mu_{U/\mathbb{R}^r \times U}$ to itself (resp. minus itself). At the level of the fiber at u, this amounts to show that the pull-back by the linear bijection

$$x' \mapsto S(u)x'$$

of $v_{\{0\}}$ is equal to itself (resp. minus itself). This follows from Exercise 1.8.8. □

Definition 2.4.5. The vector bundle E is A-*orientable* if the sheaf $or^A_{B/E}$ is constant. An A-*orientation* of E is the data of an isomorphism

$$A_B \xrightarrow{\sim} or^A_{B/E}$$

or of the corresponding A-*orientation class*

$$\mu^A_{B/E} \in \Gamma(B; or^A_{B/E}).$$

An *orientation* of the vector bundle E is the data of an orientation on every fiber E_b ($b \in B$) in such a way that for any point $b \in B$ there is a neighborhood U and a frame $(e_1, \cdots, e_r)$ of E on U with the property that the basis $(e_1(b'), \cdots, e_r(b'))$ is positively oriented in $E_{b'}$ for any $b' \in U$. The vector bundle E is *orientable* if it can be given an orientation.

Proposition 2.4.6.

(a) Any vector bundle E is canonically $\mathbb{Z}_2$-oriented.

(b) A vector bundle E is $\mathbb{Z}$-orientable if and only if it is orientable. Moreover, any $\mathbb{Z}$-orientation of E corresponds to a canonically determined orientation and vice-versa.

Proof. We use the notations of Proposition 2.4.4. Denote

$$\mu^A_{U/e} \in \Gamma(U; or^A_{B/E})$$

the section corresponding to the isomorphism ψ_e. Clearly,

$$\mu^A_{U/e'} = \pm\mu^A_{U/e}$$

if $\det S(u) \gtrless 0$ on U. Therefore, in case (a), we get $\mu^{\mathbb{Z}_2}_{U/e'} = \mu^{\mathbb{Z}_2}_{U/e}$ and there is a unique class $\mu^{\mathbb{Z}_2}_{B/E} \in \Gamma(U; or^{\mathbb{Z}_2}_{B/E})$ such that $\mu^{\mathbb{Z}_2}_{B/E_{|U}} = \mu^{\mathbb{Z}_2}_{U/e}$ for any frame e of E on U.

To prove (b), let us proceed as follows. Assume E is $\mathbb{Z}$-orientable and let $\mu_{B/E}$ be a $\mathbb{Z}$-orientation class of E. Let $b \in B$ and let e be a frame of E on a neighborhood U of b. The only generators of $\mathbb{Z}$ being 1 and -1, we know that

$$(\mu_{B/E})_b = \pm(\mu_{U/e})_b.$$

Hence, restricting U if necessary, we may assume that

$$\mu_{B/U_{|U}} = \pm\mu_{U/e}.$$

Changing the sign of one of the sections of e if necessary, we see that for any $b \in B$ there is a neighborhood U of b and a frame e of E on U such that

$$\mu_{B/E_{|U}} = \mu_{U/e}.$$

Choosing the orientations of the fibers E_b in order to make all these frames positively oriented gives us a canonical orientation of E. The reverse procedure is similar. □

2.5 Thom isomorphism and Gysin exact sequence

Definition 2.5.1. Let E be an A-oriented real vector bundle with base B. The *Thom class of* E is the image of the orientation class $\mu^A_{B/E}$ by the canonical isomorphism

$$\Gamma(B; \mathrm{or}^A_{B/E}) \simeq \mathrm{H}^r_B(E; A).$$

We denote it by τ^A_E.

Proposition 2.5.2 (Thom isomorphism). *Let E be an A-oriented real vector bundle with base B. Then,*

$$\begin{aligned} \mathrm{H}^k(B; A_B) &\to \mathrm{H}^{k+r}_B(E; A_E) \\ c^k &\mapsto \tau^A_E \cup p^*(c^k) \end{aligned}$$

is an isomorphism.

Proof. This follows directly from part (a) of Proposition 2.4.4. □

Definition 2.5.3. The *Euler class* of an A-oriented vector bundle E with base B is the class $e^A_E \in \mathrm{H}^r(B; A_B)$ defined by setting $e^A_E = ({\tau'}^A_E)_{|B}$ where ${\tau'}^A_E$ is the image of τ^A_E in $\mathrm{H}^r(E; A_E)$.

Proposition 2.5.4. *Let E be an A-oriented real vector bundle. Then, a necessary condition for the existence of a nowhere vanishing section of E is that*

$$e^A_E = 0.$$

Proof. Set $\dot{E} = E \setminus B$ and denote $s_0 : B \to E$ the zero section of E. Assume $s : B \to E$ is a nowhere vanishing section of E. Thanks to the homotopy theorem, we know that

$$s^* = s_0^*.$$

It follows that $e_E = s_0^* {\tau'}^A_E = s^* {\tau'}^A_E$. Using the inclusion $s(B) \subset \dot{E}$, we see that $e_E = s^*(({\tau'}^A_E)_{|\dot{E}})$. Since $\tau^A_E \in \mathrm{H}^r_B(E; A)$, it is clear that $({\tau'}^A_E)_{|\dot{E}} = 0$. Hence, $e^A_E = 0$ and the conclusion follows. □

Proposition 2.5.5 (Gysin exact sequence). *For any A-oriented vector bundle E with rank r and base B, there is a long exact sequence of the form*

$$\begin{tikzcd} \cdots \mathrm{H}^{k-r}(B; A) \arrow[r, "e^A_E \smile \cdot"] & \mathrm{H}^k(B; A) \arrow[r, "\dot{p}^*"] & \mathrm{H}^k(\dot{E}; A) \arrow[dll, out=0, in=180] \\ \mathrm{H}^{k+1-r}(B; A) \arrow[r, "e^A_E \smile \cdot"] & \mathrm{H}^{k+1}(B; A) \arrow[r, "\dot{p}^*"] & \mathrm{H}^{k+1}(\dot{E}; A) \cdots \end{tikzcd} \qquad (*)$$

Proof. Consider the excision distinguished triangle

$$\mathrm{R}\Gamma_B(E; A_E) \to \mathrm{R}\Gamma(E; A_E) \to \mathrm{R}\Gamma(\dot{E}; A_{\dot{E}}) \xrightarrow{+1}$$

Taking cohomology, we get the long exact sequence

$$\begin{tikzcd} \cdots \mathrm{H}^k_B(E; A_E) \arrow[r] & \mathrm{H}^k(E; A_E) \arrow[r] & \mathrm{H}^k(\dot{E}; A_E) \arrow[dll, out=0, in=180] \\ \mathrm{H}^{k+1}_B(E; A_E) \arrow[r] & \mathrm{H}^{k+1}(E; A_E) \arrow[r] & \mathrm{H}^{k+1}(\dot{E}; A_E) \cdots \end{tikzcd} \qquad (**)$$

We know by Proposition 2.5.2 that

$$\tau_E^A \smile \cdot : \mathrm{H}^{k-r}(B; A_B) \to \mathrm{H}^k_B(E; A_E)$$

is an isomorphism. Moreover, the homotopy theorem shows that

$$p^* : \mathrm{H}^k(B; A_B) \to \mathrm{H}^k(E; A_E)$$

is an isomorphism, its inverse isomorphism being

$$s_0^* : \mathrm{H}^k(E; A_E) \to \mathrm{H}^k(B; A_B).$$

Using these isomorphisms, we transform easily (**) into (*). □

Exercise 2.5.6. *Let $n \in \mathbb{N}_0$. Consider the real projective space of dimension n*

$$\mathbb{P}_n(\mathbb{R}) = \{d : d \textit{ line of } \mathbb{R}^{n+1} \textit{ through } 0\} \simeq (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^*$$

and the tautological real line bundle on $\mathbb{P}_n(\mathbb{R})$

$$\mathbb{U}_n(\mathbb{R}) = \{(v, d) \in \mathbb{R}^{n+1} \times \mathbb{P}_n(\mathbb{R}) : v \in d\}$$

the projection $p : \mathbb{U}_n(\mathbb{R}) \to \mathbb{P}_n(\mathbb{R})$ being defined by $p(v, d) = d$. Show by using a suitable Gysin exact sequence that the morphism of graded rings

$$\mathbb{Z}_2[X]/(X^{n+1}) \to \mathrm{H}^\cdot(\mathbb{P}_n(\mathbb{R}); \mathbb{Z}_2)$$

defined by sending X to $e^{\mathbb{Z}_2}_{\mathbb{U}_n(\mathbb{R})} \in \mathrm{H}^1(\mathbb{P}_n(\mathbb{R}); \mathbb{Z}_2)$ is an isomorphism.

Solution. For short, set $e = e^{\mathbb{Z}_2}_{\mathbb{U}_n(\mathbb{R})}$. Consider the Gysin exact sequence

$$\begin{aligned}
\cdots\cdots\ \mathrm{H}^{k-1}(\mathbb{P}_n(\mathbb{R}); \mathbb{Z}_2) &\xrightarrow{e\smile\cdot} \mathrm{H}^k(\mathbb{P}_n(\mathbb{R}); \mathbb{Z}_2) \xrightarrow{\dot{p}^*} \mathrm{H}^k(\dot{\mathbb{U}}_n(\mathbb{R}); \mathbb{Z}_2) \longrightarrow \\
\longrightarrow \mathrm{H}^k(\mathbb{P}_n(\mathbb{R}); \mathbb{Z}_2) &\xrightarrow{e\smile\cdot} \mathrm{H}^{k+1}(\mathbb{P}_n(\mathbb{R}); \mathbb{Z}_2) \xrightarrow{\dot{p}^*} \mathrm{H}^{k+1}(\dot{\mathbb{U}}_n(\mathbb{R}); \mathbb{Z}_2)\ \cdots\cdots
\end{aligned}$$

Clearly, the first projection $\dot{\mathbb{U}}_n(\mathbb{R}) \to \mathbb{R}^{n+1} \setminus \{0\}$ is a homeomorphism. Hence, $\mathrm{H}^k(\dot{\mathbb{U}}_n(\mathbb{R}); \mathbb{Z}_2) \simeq \mathrm{H}^k(\mathbb{R}^{n+1} \setminus \{0\}; \mathbb{Z}_2)$. Moreover, by homotopy, we know that

$$\mathrm{H}^k(\mathbb{R}^{n+1} \setminus \{0\}; \mathbb{Z}_2) \simeq \mathrm{H}^k(S_n; \mathbb{Z}_2) \simeq \begin{cases} \mathbb{Z}_2 & \text{if } k = 0, n; \\ 0 & \text{otherwise.} \end{cases}$$

Assuming $n > 1$ and denoting $q : S_n \to \mathbb{P}_n(\mathbb{R})$ the canonical map, we get the exact sequences

$$0 \to \mathrm{H}^0(\mathbb{P}_n(\mathbb{R}); \mathbb{Z}_2) \xrightarrow{q^*} \mathrm{H}^0(S_n; \mathbb{Z}_2) \to \mathrm{H}^0(\mathbb{P}_n(\mathbb{R}); \mathbb{Z}_2) \xrightarrow{e\smile\cdot} \mathrm{H}^1(\mathbb{P}_n(\mathbb{R}); \mathbb{Z}_2) \to 0$$

$$0 \to \mathrm{H}^{n-1}(\mathbb{P}_n(\mathbb{R});\mathbb{Z}_2) \xrightarrow{e\smile\cdot} \mathrm{H}^n(\mathbb{P}_n(\mathbb{R});\mathbb{Z}_2) \xrightarrow{q^*} \mathrm{H}^n(S_n;\mathbb{Z}_2) \xrightarrow{\alpha} \mathrm{H}^n(\mathbb{P}_n(\mathbb{R});\mathbb{Z}_2) \to 0$$

and isomorphisms

$$\mathrm{H}^k(\mathbb{P}_n(\mathbb{R});\mathbb{Z}_2) \xrightarrow{e\smile\cdot} \mathrm{H}^{k+1}(\mathbb{P}_n(\mathbb{R});\mathbb{Z}_2)$$

for $1 \leq k \leq n-1$. We know that $\mathbb{P}_n(\mathbb{R})$ is a compact connected topological manifold of dimension n. Therefore, $\mathrm{H}^0(\mathbb{P}_n(\mathbb{R});\mathbb{Z}_2) \simeq \mathbb{Z}_2$ and

$$q^* : \mathrm{H}^0(\mathbb{P}_n(\mathbb{R});\mathbb{Z}_2) \to \mathrm{H}^0(S_n;\mathbb{Z}_2)$$

is an isomorphism. Hence,

$$\mathrm{H}^0(\mathbb{P}_n(\mathbb{R});\mathbb{Z}_2) \xrightarrow{e\smile\cdot} \mathrm{H}^1(\mathbb{P}_n(\mathbb{R});\mathbb{Z}_2)$$

is an isomorphism. By Poincaré duality, we have

$$\mathrm{H}^n(\mathbb{P}_n(\mathbb{R});\mathbb{Z}_2) \simeq \mathrm{H}^0(\mathbb{P}_n(\mathbb{R});\mathbb{Z}_2) \simeq \mathbb{Z}_2.$$

The morphism α being surjective is thus an isomorphism. It follows that

$$\mathrm{H}^{n-1}(\mathbb{P}_n(\mathbb{R});\mathbb{Z}_2) \xrightarrow{e\smile\cdot} \mathrm{H}^n(\mathbb{P}_n(\mathbb{R});\mathbb{Z}_2)$$

is also an isomorphism. Summing up, for $n > 1$, we have established that

$$\mathrm{H}^k(\mathbb{P}_n(\mathbb{R});\mathbb{Z}_2) \xrightarrow{e\smile\cdot} \mathrm{H}^{k+1}(\mathbb{P}_n(\mathbb{R});\mathbb{Z}_2)$$

is an isomorphism for $k = 0, \cdots, n-1$. Since we have also $\mathrm{H}^0(\mathbb{P}_n(\mathbb{R});\mathbb{Z}_2) \simeq \mathbb{Z}_2$ and $\mathrm{H}^k(\mathbb{P}_n(\mathbb{R});\mathbb{Z}_2) \simeq 0$ if $k > n$, the conclusion follows easily. The case $n = 1$ is treated similarly. $\square$

2.6 Euler classes of inverse images and direct sums

Proposition 2.6.1. *Let $f : Y \to X$ be a continuous map and let E be a vector bundle on X. Then, there is a canonical isomorphism*

$$f^{-1}\, or^A_{X/E} \xrightarrow{\sim} or^A_{Y/f^{-1}(E)}\,.$$

In particular, any A-orientation of E on X induces an A-orientation of $f^{-1}(E)$ on Y.

Proof. Consider the commutative diagram

$$\begin{tikzcd} f^{-1}(E) \arrow[r, "g"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

where

$$g_{|f^{-1}(E)_y} : f^{-1}(E)_y \to E_{f(y)}$$

is the canonical isomorphism, p and q being the projections of the vector bundles E and $f^{-1}(E)$. Since g is continuous, it induces canonical pull-back morphisms

$$\mathrm{R}\Gamma_U(E_{|U}; A_E) \to \mathrm{R}\Gamma_V(f^{-1}(E)_{|V}; A_{f^{-1}(E)})$$

for any open subsets U, V of X, Y such that $V \subset f^{-1}(U)$. Taking cohomology, we obtain in the same conditions a canonical morphism

$$\Gamma(U; or^A_{X/E}) \to \Gamma(V; or^A_{Y/f^{-1}(E)}).$$

This gives us a morphism of sheaves of A-module

$$f^{-1}\, or^A_{X/E} \to or^A_{Y/f^{-1}(E)} \tag{*}$$

It remains to prove that this is an isomorphism. At the level of fibers, (*) may be visualized through the commutative diagram

$$\begin{array}{ccc}
(or^A_{X/E})_{f(y)} & \longrightarrow & (or^A_{Y/f^{-1}(E)})_y \\
\downarrow \wr & & \downarrow \wr \\
\mathrm{H}^r_{\{f(y)\}}(E_{f(y)}; A) & \xrightarrow[g^*_{|f^{-1}(E)_y}]{} & \mathrm{H}^r_{\{y\}}(f^{-1}(E)_y; A)
\end{array}$$

Since $g_{|f^{-1}(E)_y} : f^{-1}(E)_y \to E_{f(y)}$ is a homeomorphism, $g^*_{|f^{-1}(E)_y}$ is an isomorphism and the conclusion follows. □

Remark 2.6.2. In the situation of the preceding proposition, assume E is oriented. Then, it is canonically $\mathbb{Z}$-oriented and we get a canonical $\mathbb{Z}$-orientation on $f^{-1}E$. One checks easily that one can characterize the corresponding orientation by the fact that if $e_1, \dots, e_r$ is a positively oriented frame of E on U then $e_1 \circ f, \dots, e_r \circ f$ is a positively oriented frame of $f^{-1}E$ on $f^{-1}(U)$.

Proposition 2.6.3. *Let $f : Y \to X$ be a continuous map and let E be an A-oriented vector bundle of rank r on X. Assume $f^{-1}(E)$ is endowed with the A-orientation induced by that of E. Then,*

$$e^A_{f^{-1}(E)} = f^* e^A_E.$$

Proof. It follows from the proof of the preceding proposition that the image by

$$g^* : \mathrm{H}^r_X(E; A) \to \mathrm{H}^r_Y(f^{-1}(E); A)$$

of τ_E is $\tau_{f^{-1}(E)}$. Therefore, denoting $s_{0,X}$ and $s_{0,Y}$ the zero sections of E and $f^{-1}(E)$, we have

$$\begin{aligned} e^A_{f^{-1}(E)} &= s^*_{0,Y}\tau'^A_{f^{-1}(E)} = s^*_{0,Y}g^*\tau'^A_E \\ &= f^* s^*_{0,X}\tau'^A_E = f^* e^A_E. \end{aligned}$$

□

Exercise 2.6.4. *Let E be a real vector bundle of rank r on X with $e^{\mathbb{Z}_2}_E \neq 0$. Show that if E can be represented as an inverse image of a real vector bundle on S_n, then $n = r$.*

Solution. This follows directly from the preceding proposition and the fact that

$$\mathrm{H}^k(S_n;\mathbb{Z}_2) = 0$$

for $k \notin \{0, n\}$. □

Proposition 2.6.5. *Let E (resp. F) be a vector bundle of rank r (resp. s) on X. Then, there is a canonical isomorphism*

$$or^A_{X/E} \otimes_A or^A_{X/F} \xrightarrow{\sim} or^A_{X/E\oplus F}\,.$$

In particular, given A-orientations for E and F, we can construct canonically an A-orientation for $E \oplus F$.

Proof. Denote $p_E : E \oplus F \to E$, $p_F : E \oplus F \to F$ the canonical projections. For any open subset U of X, consider the pull-back morphisms

$$\mathrm{R}\Gamma_U(E_{|U}; A_E) \to \mathrm{R}\Gamma_{p_E^{-1}(U)}((E \oplus F)_{|U}; A_{E\oplus F})$$

$$\mathrm{R}\Gamma_U(F_{|U}; A_F) \to \mathrm{R}\Gamma_{p_F^{-1}(U)}((E \oplus F)_{|U}; A_{E\oplus F}).$$

Combining them with a cup product, we get a canonical morphism

$$\mathrm{R}\Gamma_U(E_{|U}; A_E) \otimes^L_A \mathrm{R}\Gamma_U(F_{|U}; A_F) \to \mathrm{R}\Gamma_U((E \oplus F)_{|U}; A_{E\oplus F})$$

since $p_E^{-1}(U) \cap p_F^{-1}(U)$ is the zero section of $(E \oplus F)_{|U}$. This gives us a canonical morphism

$$\Gamma(U; or^A_{X/E}) \otimes_A \Gamma(U; or^A_{X/F}) \to \Gamma(U; or^A_{X/E\oplus F})$$

and consequently a morphism of sheaves of A-modules

$$or^A_{X/E} \otimes_A or^A_{X/F} \to or^A_{X/E\oplus F}\,.$$

To prove that it is an isomorphism, we will work at the level of fibers. Using the commutative diagram

$$\begin{tikzcd}
(or^A_{X/E})_x \otimes_A (or^A_{X/F})_x \arrow[r] \arrow[d, "\wr"] & (or^A_{X/E\oplus F})_x \arrow[d, "\wr"] \\
\mathrm{H}^r_{\{x\}}(E_x;A) \otimes_A \mathrm{H}^s_{\{x\}}(F_x;A) \arrow[r] & \mathrm{H}^{r+s}_{\{x\}}(E_x \oplus F_x;A)
\end{tikzcd}$$

we see that it is sufficient to establish that the second horizontal arrow is an isomorphism. This follows directly from the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^r_{\{x\}}(E_x;A) \otimes_A \mathrm{H}^s_{\{x\}}(F_x;A) \arrow[r] \arrow[d, "\wr"] & \mathrm{H}^{r+s}_{\{x\}}(E_x \oplus F_x;A) \arrow[d, "\wr"] \\
\mathrm{H}^r_c(E_x;A) \otimes_A \mathrm{H}^s_c(F_x;A) \arrow[r] & \mathrm{H}^{r+s}_c(E_x \oplus F_x;A)
\end{tikzcd}$$

and Künneth theorem. (There is no torsion problem since both $\mathrm{H}^r_c(E_x;A)$ and $\mathrm{H}^s_c(F_x;A)$ are isomorphic to A.) □

Remark 2.6.6. In the situation of the preceding proposition, assume E and F are oriented. Then, the canonical $\mathbb{Z}$-orientations of E and F induce a canonical $\mathbb{Z}$-orientation of $E\oplus F$. One checks easily that one can characterize the corresponding orientation by the fact that if $e_1,\dots,e_r$ (resp. $f_1,\dots,f_r$) is a positively oriented local frames of E (resp. F) then

$$e_1,\dots,e_r,f_1,\dots,f_r$$

is a positively oriented local frame of $E \oplus F$.

Proposition 2.6.7. *Let E and F be two A-oriented vector bundles of rank r and s on X. Assume $E \oplus F$ is endowed with the A-orientation induced by that of E and F. Then,*

$$e^A_{E\oplus F} = e^A_E \smile e^A_F.$$

Proof. It follows from the proof of the preceding proposition that

$$\tau^A_{E\oplus F} = p^*_E \tau^A_E \smile p^*_F \tau^A_F.$$

Therefore, denoting $s_{0,E\oplus F}$, $s_{0,E}$ and $s_{0,F}$ the zero sections of $E \oplus F$, E and F, we get

$$\begin{aligned}
s^*_{0,E\oplus F}({\tau'}^A_{E\oplus F}) &= s^*_{0,E\oplus F} p^*_E {\tau'}^A_E \smile s^*_{0,E\oplus F} p^*_F {\tau'}^A_F \\
&= s^*_{0,E} {\tau'}^A_E \smile s^*_{0,F} {\tau'}^A_F \\
&= e^A_E \smile e^A_F.
\end{aligned}$$

□

Exercise 2.6.8. *Show that an oriented real vector bundle E on S_n with $e_E^{\mathbb{Z}} \neq 0$ has no proper sub-bundle.*

Solution. Let us proceed by contradiction. Assume F is a proper sub-bundle of rank s of E. Denote G the quotient bundle E/F and t its rank. The assumption on E combined with the cohomology table

$$\mathrm{H}^k(S_n; \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } k = 0, n \\ 0 & \text{otherwise,} \end{cases}$$

shows that the rank of E is n. It follows that $0 < s < n$ and $0 < t < n$. Since $\pi_1(S_n) \simeq 1$, F and G are orientable. Moreover, we may choose their orientations in such a way that

$$E \simeq F \oplus G$$

as oriented real vector bundles. In this case, we have $e_E^{\mathbb{Z}} = e_F^{\mathbb{Z}} \smile e_G^{\mathbb{Z}}$. Since both $e_F^{\mathbb{Z}}$ and $e_G^{\mathbb{Z}}$ are 0, we get $e_E^{\mathbb{Z}} = 0$ contrarily to what has been assumed. ☐

2.7 Euler classes of normal bundles

Proposition 2.7.1. *Let X and Y be topological manifolds of dimension d_X and d_Y. Assume Y is a closed subspace of X and denote $i : Y \to X$ the canonical inclusion. Then,*

$$i^{-1}\mathrm{R}\Gamma_Y A_X \simeq i^! A_X \simeq \mathrm{or}_Y^A \otimes_A i^{-1}(\mathrm{or}_X^A)^\vee[d_Y - d_X].$$

Proof. We know that $\omega_X^A = \mathrm{or}_X^A[d_X]$ and that $\omega_Y^A = \mathrm{or}_Y^A[d_Y]$. Since $\omega_Y^A = i^!\omega_X^A$, and since or_X^A is locally isomorphic to A_X, the conclusion follows. ☐

Definition 2.7.2. In the situation of the preceding proposition, we call

$$\mathrm{or}_{Y/X}^A = \mathrm{or}_Y^A \otimes_A i^{-1}(\mathrm{or}_X^A)^\vee$$

the *relative A-orientation sheaf of Y in X*. A *relative A-orientation of Y in X* is the datum of an isomorphism

$$A_Y \xrightarrow{\sim} \mathrm{or}_{Y/X}^A .$$

The associated *relative A-orientation class* is the image

$$\mu_{Y/X}^A \in \mathrm{H}^0(Y; \mathrm{or}_{Y/X}^A)$$

of 1_Y by this isomorphism. The *Thom class* of a relative A-orientation of Y in X is the class

$$\tau^A_{Y/X} \in \mathrm{H}^{d_X - d_Y}_Y(X; A_X)$$

corresponding to $\mu^A_{Y/X}$ through the isomorphism of the preceding proposition. The restriction of $\tau^A_{Y/X}$ to Y is called the *Euler class* $e^A_{Y/X}$ of the relative A-orientation.

Remark 2.7.3. If, in the situation considered above, X and Y are both oriented, then we have isomorphisms

$$\mathbb{Z}_X \simeq \mathrm{or}_X \quad , \quad \mathbb{Z}_Y \simeq \mathrm{or}_Y$$

and consequently an isomorphism

$$\mathbb{Z}_Y \simeq \mathrm{or}_{Y/X} .$$

We leave it to the reader to check that the Thom class of the corresponding relative orientation of Y in X is

$$i_!(1).$$

In particular, the preceding definition is compatible with Definition 1.14.14.

Proposition 2.7.4. *Let Y be a closed differential submanifold of dimension d_Y of the compact differential manifold X of dimension d_X. Then, there is a canonical isomorphism*

$$\mathrm{or}^A_{Y/X} \simeq \mathrm{or}^A_{Y/T_Y X} .$$

In particular, relative A-orientations of Y in X correspond to A-orientations of the real vector bundle $T_Y X \to Y$. Moreover, for corresponding orientations, we have

$$e^A_{Y/X} = e^A_{T_Y X}.$$

Proof. Endow X with a Riemannian metric and identify $T_Y X$ with the orthogonal complement of TY in $TX_{|Y}$. For any $\epsilon > 0$, set $V_\epsilon = \{v_y \in (T_Y X)_y : |v_y| < \epsilon\}$. As is well-known, for ϵ sufficiently small, the map

$$\varphi : V_\epsilon \to \varphi(V_\epsilon)$$

which sends $v_y \in V_\epsilon$ to $\exp_y(v_y) \in X$ is a diffeomorphism such that $\varphi(y) = y$ for any $y \in Y$. It follows that

$$\varphi^* : \mathrm{R}\Gamma_Y(A_{\varphi(V_\epsilon)}) \to \mathrm{R}\Gamma_Y(A_{V_\epsilon})$$

is an isomorphism. Since

$$\mathrm{R}\Gamma_Y(A_{\varphi(V_\epsilon)}) \simeq \mathrm{R}\Gamma_Y(A_X) \simeq \mathrm{or}^A_{Y/X}[d_Y - d_X]$$

and

$$\mathrm{R}\Gamma_Y(A_{V_\epsilon}) \simeq \mathrm{R}\Gamma_Y(A_{T_Y X}) \simeq \mathrm{or}^A_{Y/T_Y X}[d_Y - d_X],$$

φ^* induces an isomorphism

$$\psi : \mathrm{or}^A_{Y/X} \xrightarrow{\sim} \mathrm{or}^A_{Y/T_Y X} \tag{*}$$

A priori, this isomorphism depends on the chosen Riemannian metric. However, since two such metrics are homotopic, it is not difficult to see that the associated maps

$$\varphi_0 : V_{\epsilon_0} \to X \quad , \quad \varphi_1 : V_{\epsilon_1} \to X$$

become homotopic when restricted to an appropriate neighborhood V of Y in $T_Y X$. One can even request that the homotopy $h : V \times [0,1] \to X$ is such that $h(y,t) = y$ for any $y \in Y$. It is then clear that the isomorphisms

$$\psi_0 : \mathrm{or}^A_{Y/X} \simeq \mathrm{or}^A_{T_Y X/Y} \quad \text{and} \quad \psi_1 : \mathrm{or}^A_{Y/X} \simeq \mathrm{or}^A_{T_Y X/Y}$$

associated to φ_0 and φ_1 are equal. Moreover, by construction, for corresponding A-orientations, we have

$$\varphi^*((\tau^A_{Y/X})_{|\varphi(V_\epsilon)}) = (\tau^A_{T_Y X})_{|V_\epsilon}.$$

Hence,

$$e^A_{T_Y X} = [\varphi^*((\tau^A_{Y/X})_{|\varphi(V_\epsilon)}]_{|Y} = (\tau^A_{Y/X})_{|Y} = e^A_{Y/X}.$$

□

Remark 2.7.5. In the preceding proposition, assume X and Y oriented and take $A = \mathbb{Z}$. Thanks to Remark 2.7.3, there is a canonical relative orientation of Y in X. By working locally, one checks easily that this relative orientation corresponds to the orientation of $T_Y X$ obtained by quotienting the orientation of $(TX)_{|Y}$ by that of TY.

Corollary 2.7.6. *Let X be a compact oriented differential manifold. Then,*

$$\epsilon_X = e_{TX}.$$

In particular,

$$\chi(X) = \int_X e_{TX}.$$

Proof. The orientation of X induces canonical orientations on Δ and $X \times X$. If we give $T_\Delta(X \times X)$ its usual quotient orientation, Remark 2.7.5 and Proposition 2.7.4 show that

$$e_{\Delta/X\times X} = e_{T_\Delta(X\times X)}.$$

Since the canonical isomorphism

$$TX \xrightarrow{\sim} \delta^{-1}T_\Delta(X \times X)$$

defined by sending $\theta \in T_xX$ to the class of $(0,\theta) \in T_{(x,x)}(X \times X)$ modulo $T_{(x,x)}\Delta$ is compatible with the orientations, we get

$$e_{TX} = \delta^* e_{T_\Delta(X\times X)}.$$

The conclusion follows Definition 2.2.1. □

Exercise 2.7.7. *Show that the tangent bundle TS_n to an even dimensional sphere S_n has no proper sub-bundle.*

Solution. Assume E has a proper sub-bundle. It follows from Exercise 2.6.8 that $e^{\mathbb{Z}}_{TS_n} = 0$ and hence that

$$\chi(S_n) = \int e^{\mathbb{Z}}_{TS_n} = 0.$$

But for n even, we have $\chi(S_n) = 1 + (-1)^n = 2$ and we get a contradiction. □

Exercise 2.7.8. *Let $p : E \to X$ be an oriented differential real vector bundle of rank r on an oriented manifold X of dimension n. Assume $s : X \to E$ is a section of E transverse to the zero section which vanishes at some points of X. Then,*

$$Z_s = \{x \in X : s(x) = 0\}$$

is a closed oriented differential submanifold of X of codimension r and the image of

$$\tau_{Z_s/X} \in \mathrm{H}^r_{Z_s}(X;\mathbb{Z})$$

in $\mathrm{H}^r(X;\mathbb{Z})$ is the Euler class of E.

Solution. It follows from our assumptions that E has a canonical structure of oriented differential manifold. Therefore, the zero section $s_0 : X \to E$ of E induces canonical morphisms

$$s_{0!} : \mathrm{H}^k(X;\mathbb{Z}) \to \mathrm{H}^{k+r}_X(E;\mathbb{Z}) \quad (k \in \mathbb{Z}).$$

Since

$$s_{0!}(c^k) = s_{0!}(s_0^* p^* c^k) = p^* c_k \smile s_{0!}(1)$$

and $s_{0!}(1) = \tau_{X/E} = \tau_E$, these morphisms coincide with Thom isomorphisms. Therefore, we have only to show that

$$s_{0!}(e_E) = s_{0!}(\tau_{Z_s/X}) = \tau_{Z_s/E}.$$

Since

$$T_{s(X)}E \simeq (p^{-1}E)_{|s(X)}$$

as oriented vector bundles, we have

$$e_E = s^* e_{T_{s(X)}E} = s^* \tau_{s(X)/E} = s_0^* \tau_{s(X)/E};$$

the last equality coming from the fact that s_0 and s are homotopic. Therefore,

$$\begin{aligned} s_{0!}(e_E) &= s_{0!} s_0^* (\tau_{s(X)/E}) \\ &= \tau_{s(X)/E} \smile (s_{0!} 1) \\ &= \tau_{s(X)/E} \smile \tau_{X/E} \end{aligned}$$

and the conclusion follows from Proposition 1.14.15. ☐

3

Characteristic classes of real vector bundles

For the sake of simplicity, all the topological spaces in this chapter will be implicitly assumed to be paracompact.

3.1 Stiefel-Whitney classes

Lemma 3.1.1 (Leray-Hirsch). *Let $f : X \to Y$ be a proper map. Assume the classes*

$$g_1 \in \mathrm{H}^{k_1}(X; A), \cdots, g_n \in \mathrm{H}^{k_n}(X; A)$$

are such that

$$g_{1|f^{-1}(y)}, \cdots, g_{n|f^{-1}(y)}$$

form a free family of generators of $\mathrm{H}^{\cdot}(f^{-1}(y); A)$ as an A-module for any $y \in Y$. Then,

$$g_1, \cdots, g_n$$

form a free family of generators of $\mathrm{H}^{\cdot}(X; A)$ for the $\mathrm{H}^{\cdot}(Y; A)$-module structure given by the left action

$$(\alpha, \beta) \mapsto f^*(\alpha) \smile \beta.$$

Proof. Using the isomorphisms

$$\mathrm{RHom}_A(A_Y[-k], Rf(A_X)) \simeq \mathrm{RHom}_A(f^{-1}A_Y, A_X)[k] \simeq \mathrm{R}\Gamma(X; A_X)[k]$$

we associate to $g_1, \cdots, g_n$ canonical morphisms

$$h_1 : A_Y[-k_1] \to Rf(A_X), \cdots, h_n : A_Y[-k_n] \to Rf(A_X).$$

This gives us a morphism

$$h : \bigoplus_{j=1}^{n} A_Y[-k_j] \to Rf(A_X).$$

Its fiber at $y \in Y$ is the morphism

$$h_y : \bigoplus_{j=1}^{n} A[-k_j] \to [Rf(A_X)]_y \simeq \mathrm{R}\Gamma(f^{-1}(y); A)$$

associated to the cohomology classes $g_{1|f^{-1}(y)}, \cdots, g_{n|f^{-1}(y)}$. Therefore, our assumption ensures that h is an isomorphism. Applying $\mathrm{R}\Gamma(Y; \cdot)$, we see that h induces the isomorphism

$$\bigoplus_{j=1}^{n} \mathrm{R}\Gamma(Y; A_Y)[-k_j] \to \mathrm{R}\Gamma(Y; Rf(A_X)) \simeq \mathrm{R}\Gamma(X; A_X).$$

It follows that the morphism of graded A-modules

$$\bigoplus_{j=1}^{n} \mathrm{H}^{\cdot}(Y; A_Y)[-k_j] \to \mathrm{H}^{\cdot}(X; A_X)$$
$$(\alpha_1, \cdots, \alpha_n) \mapsto (p^*(\alpha_1) \smile g_1 + \cdots + p^*(\alpha_n) \smile g_n)$$

is an isomorphism, hence the conclusion. □

Let E be a vector bundle of rank r based on X. As usual, we define the real projective bundle $P(E)$ associated to E by setting

$$P(E) = \{(d, x) : x \in X, d \text{ line of } E_x \text{ trough } 0_x\},$$

the projection $\pi : P(E) \to X$ being defined by setting

$$\pi(d, x) = x.$$

Of course, $P(E) \simeq \dot{E}/\mathbb{R}^*$ where the action of $\mathbb{R}^*$ on $\dot{E}$ is defined fiberwise using the real vector space structure of E_x ($x \in X$). As is well-known, $P(E)$

has a canonical topology. For this topology, π is continuous and using local frames of E, we see that for any $x \in X$ there is a neighborhood U of x and a commutative diagram of the form

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & & \mathbb{P}_{r-1}(\mathbb{R}) \times U \arrow[dl, "p_U"] \\ & U & \end{tikzcd} \tag{*}$$

In particular, π is proper and its fibers are homeomorphic to $\mathbb{P}_{r-1}(\mathbb{R})$. Recall that

$$E \times_X P(E) = \{(e, (d, x)) : p(e) = x\}$$

and denote $U(E)$ the subset of $E \times_X P(E)$ defined by

$$U(E) = \{(e, (d, x)) : p(e) = x, e \in d\}.$$

One checks easily that the second projection

$$p_{P(E)} : U(E) \to P(E)$$

turns $U(E)$ into a line bundle with base $P(E)$. By construction, a homeomorphism

$$\pi^{-1}(U) \simeq \mathbb{P}_{r-1}(\mathbb{R}) \times U$$

of the type (*) transforms $U(E)_{|\pi^{-1}(U)}$ into a line bundle isomorphic to

$$p^{-1}_{\mathbb{P}_{r-1}(\mathbb{R})} \mathbb{U}_{r-1}(\mathbb{R}).$$

Proposition 3.1.2. *Let E be a vector bundle of rank r on X. Set*

$$\xi = e^{\mathbb{Z}_2}_{U(E)} \in \mathrm{H}^1(P(E); \mathbb{Z}_2).$$

Then,

$$1, \xi, \cdots, \xi^{r-1}$$

form a free family of generators of $\mathrm{H}^{\cdot}(P(E); \mathbb{Z}_2)$ as an $\mathrm{H}^{\cdot}(X; \mathbb{Z}_2)$-module. In other words, any $\beta \in \mathrm{H}^{\cdot}(P(E); \mathbb{Z}_2)$ may be written in a unique way as

$$\beta = \pi^* \alpha_0 + \pi^* \alpha_1 \smile \xi + \cdots + \pi^* \alpha_{r-1} \smile \xi^{r-1}$$

with $\alpha_0, \cdots, \alpha_{r-1} \in \mathrm{H}^{\cdot}(X; \mathbb{Z}_2)$.

Proof. We know from Exercise 2.5.6 that the morphism

$$\mathbb{Z}_2[X]/(X^r) \to \mathrm{H}^{\cdot}(\mathbb{P}_{r-1}(\mathbb{R}); \mathbb{Z}_2)$$

which sends X to $e = e^{\mathbb{Z}_2}_{\mathbb{U}_{r-1}(\mathbb{R})}$ is an isomorphism. It follows that

$$1, e, \cdots, e^{r-1}$$

is a free family of generators of $\mathrm{H}^{\cdot}(\mathbb{P}_{r-1}(\mathbb{R}); \mathbb{Z}_2)$ as a $\mathbb{Z}_2$-module. Since $\xi_{|\pi^{-1}(x)}$ corresponds to e through the canonical isomorphism

$$U(E)_{|p^{-1}_{P(E)}(x)} \simeq \mathbb{U}_{r-1}(\mathbb{R}),$$

we see that

$$1, \xi_{|\pi^{-1}(x)}, \cdots, \xi^{r-1}_{|\pi^{-1}(x)}$$

form a free family of generators of $\mathrm{H}^{\cdot}(\pi^{-1}(x); \mathbb{Z}_2)$ for any $x \in X$ and the conclusion follows from Lemma 3.1.1. □

Definition 3.1.3. Let E be a vector bundle of rank r based on the space X. Using the preceding proposition, we define the *Stiefel-Whitney classes* $w_1(E), \cdots, w_r(E)$ of E by the formula

$$\xi^r = \pi^*(w_r(E)) + \pi^*(w_{r-1}(E)) \smile \xi + \cdots + \pi^*(w_1(E)) \smile \xi^{r-1}.$$

We also set $w_0(E) = 1$ and $w_k(E) = 0$ for $k > r$. By construction,

$$w_k(E) \in \mathrm{H}^k(X; \mathbb{Z}_2)$$

for any $k \in \mathbb{N}$. We also define the *total Stiefel-Whitney class* $w(E)$ of E by the formula

$$w(E) = w_0(E) + \cdots + w_r(E) \in \mathrm{H}^{\cdot}(X; \mathbb{Z}_2).$$

Proposition 3.1.4. *Let $f : Y \to X$ be a morphism of topological spaces. Assume E is a vector bundle of rank r on X. Then,*

$$w(f^{-1}(E)) = f^* w(E).$$

Proof. Consider the canonical diagram

$$\begin{array}{ccc} f^{-1}(E) & \xrightarrow{g} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[f]{} & X \end{array}$$

Define $P(g) : P(f^{-1}(E)) \to P(E)$ as the map

$$(d, y) \mapsto (g(d), f(y))$$

and $U(g) : U(f^{-1}(E)) \to U(E)$ as the map

$$(e,(d,y)) \mapsto (g(e),(g(d),f(y))).$$

We get the commutative diagram

$$\begin{array}{ccc} U(f^{-1}(E)) & \xrightarrow{U(g)} & U(E) \\ \downarrow & & \downarrow \\ P(f^{-1}(E)) & \xrightarrow{P(g)} & P(E) \\ \varpi\downarrow & & \downarrow\pi \\ Y & \xrightarrow[f]{} & X \end{array}$$

Since $g_{|f^{-1}(E)_y} : f^{-1}(E)_y \to E_{f(y)}$ is an isomorphism, so is

$$U(g)_{|U(f^{-1}(E))_{(d,y)}} : U(f^{-1}(E))_{(d,y)} \to U(E)_{(g(d),f(y))}.$$

It follows that $U(f^{-1}(E)) \simeq P(g)^{-1}U(E)$ and hence that

$$\xi_{f^{-1}(E)} = e^{\mathbb{Z}_2}_{U(f^{-1}(E))} = P(g)^* e^{\mathbb{Z}_2}_{U(E)} = P(g)^* \xi_E.$$

Applying $P(g)^*$ to the relation

$$\xi_E^r = \pi^*(w_r(E)) + \pi^*(w_{r-1}(E)) \smile \xi_E + \cdots + \pi^*(w_1(E)) \smile \xi_E^{r-1},$$

we get

$$\begin{aligned} &\xi^r_{f^{-1}(E)} \\ &\qquad = \varpi^* f^*(w_r(E)) + \varpi^* f^*(w_{r-1}(E)) \smile \xi_{f^{-1}(E)} + \cdots \\ &\qquad\quad + \varpi^* f^*(w_1(E)) \smile \xi^{r-1}_{f^{-1}(E)} \end{aligned}$$

and the conclusion follows from the definition of the Stiefel-Whitney classes of $f^{-1}(E)$. □

Proposition 3.1.5. *Let E, F be vector bundles of rank r and s on the topological space X. Then,*

$$w(E \oplus F) = w(E) \smile w(F).$$

Proof. Consider the map

$$i : E \to E \oplus F$$

defined at the level of fibers by setting

$$i(e_x) = (e_x, 0)$$

and denote $j : P(E) \to P(E \oplus F)$ the map induced by i. Consider also the map

$$p : E \oplus F \to F$$

defined at the level of fibers by setting

$$p_x(e_x, f_x) = f_x$$

and denote

$$q : P(E \oplus F) \setminus j(P(E)) \to P(F)$$

the map induced by p. Finally, denote

$$\rho : P(E \oplus F) \setminus j(P(E)) \to P(E \oplus F)$$

the inclusion map. One checks easily that

$$j^{-1}U(E \oplus F) \simeq U(E)$$

and that

$$\rho^{-1}U(E \oplus F) \simeq q^{-1}U(F).$$

Set

$$\alpha = \sum_{k=0}^{r} \pi^*_{E\oplus F}(w_k(E))\xi^{r-k}_{E\oplus F}$$

and

$$\beta = \sum_{l=0}^{s} \pi^*_{E\oplus F}(w_l(F))\xi^{s-l}_{E\oplus F}.$$

Clearly, we have

$$\begin{aligned} j^*\alpha &= \sum_{k=0}^{r} j^*\pi^*_{E\oplus F}(w_k(E))j^*\xi^{r-k}_{E\oplus F} \\ &= \sum_{k=0}^{r} \pi^*_E(w_k(E))\xi^{r-k}_E \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \rho^*\beta &= \sum_{l=0}^{s} \rho^*\pi^*_{E\oplus F}(w_l(F))\rho^*\xi^{s-l}_{E\oplus F} \\ &= \sum_{l=0}^{s} q^*\pi^*_F(w_l(F))q^*\xi^{s-l}_F \\ &= q^*\left(\sum_{l=0}^{s} \pi^*_F(w_l(F))\xi^{s-l}_F\right) \\ &= 0. \end{aligned}$$

From the second relation, it follows that $\beta = \sigma(\gamma)$ where

$$\sigma : \mathrm{H}^{\cdot}_{j(P(E))}(P(E \oplus F); \mathbb{Z}_2) \to \mathrm{H}^{\cdot}(P(E \oplus F); \mathbb{Z}_2)$$

is the canonical morphism. Using the fact that for

$$\smile: \mathrm{H}^{\cdot}(j(P(E)); \mathbb{Z}_2) \otimes \mathrm{H}^{\cdot}_{j(P(E))}(P(E \oplus F); \mathbb{Z}_2) \to \mathrm{H}^{\cdot}_{j(P(E))}(P(E \oplus F); \mathbb{Z}_2)$$

we have

$$\alpha \smile \sigma(\gamma) = \sigma(j^*\alpha \smile \gamma),$$

we see that

$$\alpha \smile \beta = 0$$

in $\mathrm{H}^{\cdot}(P(E \oplus F); \mathbb{Z}_2)$. It follows that

$$\sum_{k=0}^{r} \sum_{l=0}^{s} \pi^*_{E \oplus F}(w_k(E) \smile w_l(F)) \xi_{E \oplus F}^{r+s-k-l} = 0$$

and hence that

$$\sum_{n=0}^{r+s} \pi^*_{E \oplus F} \left(\sum_{k=0}^{n} w_k(E) \smile w_{n-k}(F) \right) \xi_{E \oplus F}^{r+s-n} = 0.$$

This shows that

$$\xi_{E \oplus F}^{r+s} = \sum_{n=1}^{r+s} \pi^*_{E \oplus F} \left([w(E) \smile w(F)]_n \right) \xi_{E \oplus F}^{r+s-n}$$

and the conclusion follows. ☐

Exercise 3.1.6. *Let E be a vector bundle of rank r on the topological space X. Show that a necessary condition for E to have a trivial sub-bundle of rank s is that*

$$w_r(E) = \cdots = w_{r-s+1}(E) = 0.$$

In particular, a necessary condition for E to be trivial is that

$$w(E) = 1.$$

Solution. Assume first that E is trivial. It follows that

$$E \simeq \mathbb{R}^r \times X \simeq a_X^{-1}(\mathbb{R}^r)$$

where $a_X : X \to \{\mathrm{pt}\}$ is the canonical map. Hence,

$$w(E) = a_X^* w(\mathbb{R}^r) = a_X^* w_0(\mathbb{R}^r) = 1.$$

Assume now that E has a trivial sub-bundle T of rank s. From the isomorphism

$$E \simeq T \oplus E/T$$

we deduce that

$$w(E) = w(T) \smile w(E/T) = w(E/T).$$

Since E/T has rank $r - s$, the conclusion follows. $\square$

3.2 Splitting principle and consequences

Proposition 3.2.1 (Splitting principle). *Let E be a real vector bundle of rank r on the space X. Then, there is a proper map*

$$f : Y \to X$$

for which the canonical map

$$f^* : \mathrm{H}^{\cdot}(X; \mathbb{Z}_2) \to \mathrm{H}^{\cdot}(Y; \mathbb{Z}_2)$$

is injective and such that

$$f^{-1}(E) \simeq L_1 \oplus \cdots \oplus L_r$$

with $L_1, \cdots, L_r$ real line bundles on Y.

Proof. Let us proceed by induction on r. For $r = 1$, the result is obvious. Assume it is true for $r - 1$ and let us prove it for r. We know that

$$\pi : P(E) \to X$$

is a proper map for which

$$\pi^* : \mathrm{H}^{\cdot}(X; \mathbb{Z}_2) \to \mathrm{H}^{\cdot}(P(E); \mathbb{Z}_2)$$

is injective. Using the canonical inclusion

$$U(E) \to E \times_X P(E) \simeq \pi^{-1}(E)$$

we see that $U(E)$ is a sub-bundle of $\pi^{-1}(E)$. Setting $F = \pi^{-1}(E)/U(E)$, we get the isomorphism

$$\pi^{-1}E \simeq U(E) \oplus F.$$

By the induction hypothesis, there is a proper map

$$g : Y \to P(E)$$

for which

$$g^* : \mathrm{H}^{\cdot}(P(E); \mathbb{Z}_2) \to \mathrm{H}^{\cdot}(Y; \mathbb{Z}_2)$$

is injective and such that

$$g^{-1}(F) \simeq L_1 \oplus \cdots \oplus L_{r-1}$$

with $L_1, \cdots, L_r$ line bundles on Y. Setting $f = \pi \circ g$ and $L_r = g^{-1}(U(E))$ allows us to conclude. □

Remark 3.2.2. As shown hereafter, the preceding result allows us to reduce the proof of formulas concerning Stiefel-Whitney classes of vector bundles to the case of line bundles.

Proposition 3.2.3. *Let E be a vector bundle of rank r on X. Then,*

$$w_r(E) = e_E^{\mathbb{Z}_2}$$

in $\mathrm{H}^r(X; \mathbb{Z}_2)$.

Proof. By the preceding proposition, we may assume that there is a proper map

$$f : Y \to X$$

with

$$f^* : \mathrm{H}^{\cdot}(X; \mathbb{Z}_2) \to \mathrm{H}_{\cdot}(Y; \mathbb{Z}_2)$$

injective and such that

$$f^{-1}(E) = L_1 \oplus \cdots \oplus L_r.$$

For such a map, we have

$$f^*(w(E)) = w(f^{-1}(E)) = w(L_1) \smile \cdots \smile w(L_r).$$

Moreover, $w(L_k) = 1 + w_1(L_k)$, hence

$$f^*(w_r(E)) = w_1(L_1) \smile \cdots \smile w_1(L_r).$$

Since

$$f^*(e_E) = e^{\mathbb{Z}_2}_{f^{-1}(E)} = e^{\mathbb{Z}_2}_{L_1} \smile \cdots \smile e^{\mathbb{Z}_2}_{L_r}$$

the proof is reduced to the case where $r = 1$. When E is a line bundle, $\pi : P(E) \to X$ is a homeomorphism and $\pi^{-1}E \simeq U(E)$. It follows that $\xi_E = \pi^* e_E$. By the definition of Stiefel-Whitney classes, this shows that $w_1(E) = e_E$, hence the conclusion. □

Remark 3.2.4 (Symmetric Polynomials). Let A be a ring with unit. Recall that the elementary symmetric polynomials of $A[X_1, \cdots, X_n]$ are the polynomials $S_{n,1}, \cdots, S_{n,n}$ characterized by the fact that

$$(X - X_1) \cdots (X - X_n) = \sum_{k=0}^{n} (-1)^k S_{n,k}(X_1, \cdots, X_n) X^{n-k}$$

in $A[X_1, \cdots, X_n, X]$. In particular, $S_{n,1} = X_1 + \cdots + X_n$ and $S_{n,n} = X_1 \cdots X_n$. Recall also that any symmetric polynomial P of degree r of $A[X_1, \cdots, X_n]$ may be written in a unique way as $Q(S_{n,1}, \cdots, S_{n,n})$ with Q in $A[X_1, \cdots, X_n]$. An algorithm to find Q by induction on n and on the degree of P is the following:

(a) Write $P(X_1, \cdots, X_{n-1}, 0)$ as $Q_1(S_{n-1,1}, \cdots, S_{n-1,n-1})$.

(b) Find the polynomial $R_1(X_1, \cdots, X_n)$ such that

$$P(X_1, \cdots, X_n) - Q_1(S_{n,1}, \cdots, S_{n,n-1}) = S_{n,n} R_1(X_1, \cdots, X_n).$$

(c) Express $R_1(X_1, \cdots, X_n)$ as $Q_2(S_{n,1}, \cdots, S_{n,n})$.

(d) Write P as

$$Q_1(S_{n,1}, \cdots, S_{n,n-1}) + S_{n,n} Q_2(S_{n,1}, \cdots, S_{n,n}).$$

Exercise 3.2.5. *Write*

$$P(X_1, X_2, X_3) = X_1 X_3^2 + X_1 X_2^2 + X_2 X_1^2 + X_2 X_3^2 + X_3 X_1^2 + X_3 X_2^2$$

as a polynomial in $S_{3,1} = X_1 + X_2 + X_3$, $S_{3,2} = X_1X_2 + X_1X_3 + X_2X_3$ *and* $S_{3,3} = X_1X_2X_3$.

Solution. Consider the polynomial

$$X_1 X_2^2 + X_2 X_1^2.$$

Clearly,

$$X_1 X_2^2 + X_2 X_1^2 = X_1 X_2 (X_1 + X_2) = S_{2,1} S_{2,2}.$$

We have

$$\begin{aligned} P - S_{3,1} S_{3,2} &= P - (X_1 + X_2 + X_3)(X_1X_2 + X_1X_3 + X_2X_3) \\ &= P - X_1^2 X_2 - X_1^2 X_3 - X_1X_2X_3 - X_1 X_2^2 - X_1X_2X_3 \\ &\qquad - X_2^2 X_3 - X_1X_2X_3 - X_1 X_3^2 - X_2 X_3^2 \\ &= -3 X_1 X_2 X_3 = -3 S_{3,3}. \end{aligned}$$

Therefore,

$$P = S_{3,1}S_{3,2} - 3S_{3,3}.$$

□

Proposition 3.2.6. *Let P be a symmetric polynomial of $\mathbb{Z}_2[Z_1, \cdots, Z_r]$. Then, to any vector bundle E on X of rank r is canonically associated a class*

$$w_P(E) \in \mathrm{H}^{\cdot}(X; \mathbb{Z}_2).$$

This class is characterized by the fact that

(a)
$$w_P(f^{-1}(E)) = f^* w_P(E)$$
for any continuous map $f : Y \to X$;

(b)
$$w_P(E) = P(e^{\mathbb{Z}_2}_{L_1}, \cdots, e^{\mathbb{Z}_2}_{L_r})$$
if $E = L_1 \oplus \cdots \oplus L_r$ with $L_1, \cdots, L_r$ line bundles.

Proof. Write

$$P(Z_1, \cdots, Z_r) = Q(S_{r,1}, \cdots, S_{r,r})$$

where $Q \in \mathbb{Z}_2[Z_1, \cdots, Z_r]$ and $S_{r,1}, \cdots, S_{r,r}$ are the elementary symmetric polynomials in r unknowns. Set

$$w_P(E) = Q(w_1(E), \cdots, w_r(E)).$$

Clearly, if $f : Y \to X$ is a continuous map, we have

$$\begin{aligned} f^* w_P(E) &= f^* Q(w_1(E), \cdots, w_r(E)) \\ &= Q(w_1(f^{-1}(E)), \cdots, w_r(f^{-1}(E))) \\ &= w_P(f^{-1}(E)). \end{aligned}$$

Moreover, if $E = L_1 \oplus \cdots \oplus L_r$ with $L_1, \cdots, L_r$ line bundles, we get

$$w(E) = (1 + \xi_1) \cdots (1 + \xi_r)$$

with $\xi_1 = e^{\mathbb{Z}_2}_{L_1}, \cdots, \xi_r = e^{\mathbb{Z}_2}_{L_r}$. It follows that

$$w(E) = 1 + S_{r,1}(\xi_1, \cdots, \xi_r) + \cdots + S_{r,r}(\xi_1, \cdots, \xi_r)$$

and hence that

$$w_k(E) = S_{r,k}(\xi_1, \cdots, \xi_r).$$

Therefore,

$$\begin{aligned} w_P(E) &= Q(w_1(E), \cdots, w_r(E)) \\ &= Q(S_{r,1}(\xi_1, \cdots, \xi_r), \cdots, S_{r,r}(\xi_1, \cdots, \xi_r)) \\ &= P(\xi_1, \cdots, \xi_r). \end{aligned}$$

The fact that these two properties characterize $w_P(\cdot)$ uniquely follows from the splitting principle. ☐

Definition 3.2.7. Let $E \to B$, $F \to B$ be two real vector bundles of rank r and s. Recall that $E \otimes F$ denotes the real vector bundle of rank rs defined by setting

$$E \otimes F = E_b \otimes F_b$$

the frame of $E \otimes F$ on U associated to a frame $e_1, \cdots, e_r$ of $E_{|U}$ and a frame $f_1, \cdots, f_s$ of $F_{|U}$ being given by the sections

$$e_i \otimes f_j \quad (i \in \{1, \dots r\}, j \in \{1, \dots s\}).$$

Proposition 3.2.8. *Let $p_1 : L_1 \to B$ and $p_2 : L_2 \to B$ be two real line bundles. Then,*

$$e^{\mathbb{Z}_2}_{L_1 \otimes L_2} = e^{\mathbb{Z}_2}_{L_1} + e^{\mathbb{Z}_2}_{L_2}.$$

Proof. Denote $\mu : L_1 \oplus L_2 \to L_1 \otimes L_2$ the continuous map defined at the level of fibers by setting

$$\mu_b(e_1, e_2) = e_1 \otimes e_2$$

and denote $q_1 : L_1 \oplus L_2 \to L_1$, $q_2 : L_1 \oplus L_2 \to L_2$ the two projections. By homotopy, we have

$$\mathrm{R}\Gamma_{L_1}(L_1 \oplus L_2; \mathbb{Z}_2) \simeq \mathrm{R}\Gamma_B(L_2; \mathbb{Z}_2)$$

and

$$\mathrm{R}\Gamma_{L_2}(L_1 \oplus L_2; \mathbb{Z}_2) \simeq \mathrm{R}\Gamma_B(L_1; \mathbb{Z}_2).$$

Hence, we deduce from the Mayer-Vietoris distinguished triangle

$$\mathrm{R}\Gamma_{L_1 \cap L_2}(L_1 \oplus L_2; \mathbb{Z}_2) \to \mathrm{R}\Gamma_{L_1}(L_1 \oplus L_2; \mathbb{Z}_2) \oplus \mathrm{R}\Gamma_{L_2}(L_1 \oplus L_2; \mathbb{Z}_2) \to \mathrm{R}\Gamma_{L_1 \cup L_2}(L_1 \oplus L_2; \mathbb{Z}_2) \xrightarrow{+1}$$

a canonical morphism

$$\mathrm{H}^1_B(L_1; \mathbb{Z}_2) \oplus \mathrm{H}^1_B(L_2; \mathbb{Z}_2) \to \mathrm{H}^1_{L_1 \cup L_2}(L_1 \oplus L_2; \mathbb{Z}_2). \qquad (*)$$

Since $\mu^{-1}(B) = L_1 \cup L_2$, we also have an isomorphism

$$\mathrm{R}\Gamma_{L_1 \cup L_2}(L_1 \oplus L_2; \mathbb{Z}_2) \simeq \mathrm{R}\Gamma_B(R\mu(\mathbb{Z}_2))$$

and hence a morphism

$$\mathrm{H}^1_B(L_1 \otimes L_2; \mathbb{Z}_2) \to \mathrm{H}^1_{L_1 \cup L_2}(L_1 \oplus L_2; \mathbb{Z}_2). \tag{**}$$

Let us prove that the image of the Thom class of $L_1 \otimes L_2$ by (**) is the image of the Thom classes of L_1 and L_2 by (*). This will give the result thanks to the commutative diagrams

$$\begin{tikzcd}
\mathrm{H}^1(B;\mathbb{Z}_2) \arrow[r, "1"] & \mathrm{H}^1(B;\mathbb{Z}_2) \\
\mathrm{H}^1(L_1 \otimes L_2;\mathbb{Z}_2) \arrow[u, "|B"] \arrow[r, "\mu^*"] & \mathrm{H}^1(L_1 \oplus L_2;\mathbb{Z}_2) \arrow[u, "|B"'] \\
\mathrm{H}^1_B(L_1 \otimes L_2;\mathbb{Z}_2) \arrow[u] \arrow[r, "\mu^*"'] & \mathrm{H}^1_{L_1 \cup L_2}(L_1 \oplus L_2;\mathbb{Z}_2) \arrow[u]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{H}^1(B;\mathbb{Z}_2) \oplus \mathrm{H}^1(B;\mathbb{Z}_2) \arrow[r, "(1\ 1)"] & \mathrm{H}^1(B;\mathbb{Z}_2) \\
\mathrm{H}^1(L_1;\mathbb{Z}_2) \oplus \mathrm{H}^1(L_2;\mathbb{Z}_2) \arrow[u, "|B"] \arrow[r, "(q_1^*\ q_2^*)"] & \mathrm{H}^1(L_1 \oplus L_2;\mathbb{Z}_2) \arrow[u, "|B"'] \\
\mathrm{H}^1_B(L_1;\mathbb{Z}_2) \oplus \mathrm{H}^1_B(L_2;\mathbb{Z}_2) \arrow[u] \arrow[r, "(q_1^*\ q_2^*)"'] & \mathrm{H}^1_{L_1 \cup L_2}(L_1 \oplus L_2;\mathbb{Z}_2) \arrow[u]
\end{tikzcd}$$

First, note that it follows from Thom isomorphism that

$$U \mapsto \mathrm{H}^1_U(L_{1|U}; \mathbb{Z}_2); \qquad U \mapsto \mathrm{H}^1_U(L_{2|U}; \mathbb{Z}_2)$$

are sheaves on B. Moreover, note also that since

$$\mathrm{H}^0_{L_{1|U} \cup L_{2|U}}(L_{1|U} \oplus L_{2|U}; \mathbb{Z}_2) = 0$$

for any open subset U of B,

$$U \mapsto \mathrm{H}^1_{L_{1|U} \cup L_{2|U}}(L_{1|U} \oplus L_{2|U}; \mathbb{Z}_2)$$

is also a sheaf on B. From these remarks, it follows that our problem is of local nature and it is sufficient to work at the level of fibers and to show that the image of the appropriate Thom classes by canonical maps

$$\mathrm{H}^1_{\{b\}}((L_1)_b; \mathbb{Z}_2) \oplus \mathrm{H}^1_{\{b\}}((L_2)_b; \mathbb{Z}_2) \to \mathrm{H}^1_{(L_1)_b \cup (L_2)_b}((L_1)_b \oplus (L_2)_b; \mathbb{Z}_2)$$

and

$$\mathrm{H}^1_{\{b\}}((L_1)_b \otimes (L_2)_b; \mathbb{Z}_2) \to \mathrm{H}^1_{(L_1)_b \cup (L_2)_b}((L_1)_b \oplus (L_2)_b; \mathbb{Z}_2)$$

corresponding to (*) and (**) are equal. Denote $\mu : \mathbb{R}^2 \to \mathbb{R}$ the multiplication, τ the generator of $\mathrm{H}^1_{\{0\}}(\mathbb{R};\mathbb{Z}_2)$ and q_1, q_2 the two projections from $\mathbb{R}^2$ to $\mathbb{R}$. Using local frames for L_1 and L_2, we are reduced to show that

$$\mu^*(\tau) = q_1^*(\tau) + q_2^*(\tau) \tag{***}$$

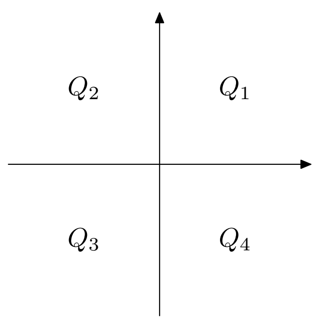
in $\mathrm{H}^1_{(\mathbb{R}\times\{0\})\cup(\{0\}\times\mathbb{R})}(\mathbb{R}^2;\mathbb{Z}_2)$. We know that τ is the image of $\chi_{]0,+\infty[}$ by the canonical map

$$\mathrm{H}^0_{\mathbb{R}\setminus\{0\}}(\mathbb{R};\mathbb{Z}_2) \to \mathrm{H}^1_{\{0\}}(\mathbb{R};\mathbb{Z}_2).$$

It follows that $\mu^*(\tau)$ (resp. $q_1^*(\tau)$, $q_2^*(\tau)$) is the image of $\chi_{Q_1} + \chi_{Q_3}$ (resp. $\chi_{Q_1} + \chi_{Q_4}$, $\chi_{Q_1} + \chi_{Q_2}$) by the canonical map

$$\mathrm{H}^0_{\mathbb{R}^2\setminus(\mathbb{R}\times\{0\})\cup(\{0\}\times\mathbb{R})}(\mathbb{R}^2;\mathbb{Z}_2) \to \mathrm{H}^1_{(\mathbb{R}\times\{0\})\cup(\{0\}\times\mathbb{R})}(\mathbb{R}^2;\mathbb{Z}_2).$$

Here, $Q_1, \cdots, Q_4$ are the open quarter planes defined in the following figure



To establish (***), it is sufficient to note that

$$\chi_{Q_1} + \chi_{Q_4} + \chi_{Q_1} + \chi_{Q_2} = \chi_{Q_2} + \chi_{Q_4} = \chi_{Q_1} + \chi_{Q_3} + \chi_{\mathbb{R}^2}$$

modulo 2. ☐

Corollary 3.2.9. *Let E be the a real vector bundle of rank r on B. Denote $\mathrm{Det}(E)$ the determinant bundle associated to E. Then,*

$$w_1(E) = e^{\mathbb{Z}_2}_{\mathrm{Det}\, E}.$$

Proof. Thanks to the splitting principle, we may assume that

$$E \simeq L_1 \oplus \ldots \oplus L_r$$

where $L_1, \ldots, L_r$ are real line bundles on B. Denote $\xi_1, \ldots, \xi_r$ their Euler classes modulo 2. Since

$$w(E) = w(L_1) \ldots w(L_r) = (1 + \xi_1) \ldots (1 + \xi_n),$$

we see that

$$w_1(E) = \xi_1 + \cdots + \xi_n.$$

Since moreover

$$\mathrm{Det}(E) \simeq \mathrm{Det}(L_1) \otimes \ldots \otimes \mathrm{Det}(L_r) \simeq L_1 \otimes \ldots \otimes L_r,$$

the preceding proposition shows that

$$e^{\mathbb{Z}_2}_{\mathrm{Det}(E)} = e^{\mathbb{Z}_2}_{L_1} + \cdots + e^{\mathbb{Z}_2}_{L_r}$$

and the conclusion follows. □

Lemma 3.2.10. *For any $r, s \in \mathbb{N}_0$ there is a unique polynomial*

$$T_{r,s} \in \mathbb{Z}[U_1, \ldots, U_r, V_1, \ldots V_s]$$

such that

$$\prod_{i=1}^{r} \prod_{j=1}^{s} (1 + X_i + Y_j) = T_{r,s}(S_{r,1}(X), \ldots, S_{r,r}(X), S_{s,1}(Y), \ldots, S_{s,s}(Y).$$

Proof. This follows directly from the fact that

$$\prod_{i=1}^{r} \prod_{j=1}^{s} (1 + X_i + Y_j)$$

is symmetric in the X and Y variables. □

Exercise 3.2.11. *Compute $T_{1,s}$ and $T_{2,2}$ explicitly.*

Solution. Since

$$(1 + X_1 + Y_1) \ldots (1 + X_1 + Y_s) = \sum_{k=0}^{s} (1 + X_1)^k S_{s,k}(Y_1, \ldots, Y_s)$$

and

$$(1 + X_1)^k = \sum_{l=0}^{k} C^l_k X^l_1,$$

we see directly that

$$T_{1,s}(U, V) = \sum_{k=0}^{s} \sum_{l=0}^{k} C^l_k U^l_1 V_k.$$

The computation of $T_{2,2}$ is also easy but more tedious. The final answer is

$$\begin{aligned}
&1 + 2U_1 + 2U_2 + 2V_1 + 2V_2 \\
&\quad + U_1^2 + 2U_1U_2 + U_2^2 + 3U_1V_1 + 2U_2V_1 + 2U_1V_2 - 2U_2V_2 + V_1^2 + 2V_1V_2 + V_2^2 \\
&\quad + U_1^2V_1 + U_1U_2V_1 + U_1^2V_2 + U_1V_1V_2 + U_1V_1^2 + U_2V_1^2
\end{aligned}$$

□

Proposition 3.2.12. *Assume E (resp. F) is a real vector bundle of rank r (resp. s) on B. Then,*

$$w(E \otimes F) = T_{r,s}(w_1(E), \dots, w_r(E), w_1(F), \dots, w_s(F)).$$

Proof. Thanks to the splitting principle, we may assume that

$$E \simeq L_1 \oplus \dots \oplus L_r \quad \text{and} \quad F \simeq N_1 \oplus \dots \oplus N_s$$

where $L_1, \dots, L_r$ and $N_1, \dots, N_s$ are real line bundles on X. Denote $\xi_1, \dots, \xi_r$ and $\eta_1, \dots, \eta_s$ their Euler classes modulo 2. Since

$$E \otimes F \simeq \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{s} L_i \otimes N_j,$$

we have

$$w(E \oplus F) = \prod_{i=1}^{r} \prod_{j=1}^{s} w(L_i \otimes N_j).$$

Hence, using Proposition 3.2.8, we get

$$w(E \oplus F) = \prod_{i=1}^{r} \prod_{j=1}^{s} (1 + \xi_i + \eta_j)$$

and the conclusion follows from the preceding lemma. □

Exercise 3.2.13. *Let E and F be real plane bundles on B. Show that*

$$\begin{aligned}
w_1(E \otimes F) &= 0 \\
w_2(E \otimes F) &= w_1(E)^2 + w_1(E)w_1(F) + w_1(F)^2 \\
w_3(E \otimes F) &= w_1(E)^2w_1(F) + w_1(E)w_1(F)^2 \\
w_4(E \otimes F) &= w_2(E)^2 + w_1(E)w_2(E)w_1(F) + w_1(E)^2w_2(F) \\
&\qquad + w_1(E)w_1(F)w_2(F) + w_2(F)^2 + w_2(E)w_1(F)^2
\end{aligned}$$

Solution. This follows by reducing modulo 2 the polynomial $T_{2,2}$ computed in Exercise 3.2.11. □

Definition 3.2.14. Let $E \to B$, $F \to B$ be two real vector bundles of rank r and s. Recall that $\operatorname{Hom}(E, F)$ denotes the real vector bundle of rank rs defined by setting

$$\operatorname{Hom}(E, F) = \operatorname{Hom}(E_b, F_b)$$

the local frame of $\operatorname{Hom}(E, F)$ on U associated to a frame $e_1, \cdots, e_r$ of $E_{|U}$ and a frame $f_1, \cdots, f_s$ of $F_{|U}$ being given by the morphisms

$$h_{ij} : E_{|U} \to F_{|U}$$

characterized by the fact that

$$h_{ji}(e_k) = \begin{cases} 0 & \text{if } k \neq i \\ f_j & \text{if } k = i \end{cases}$$

If F is the trivial bundle $\mathbb{R} \times B \to B$, then $\operatorname{Hom}(E, F)$ is simply denoted E^* and is called the *dual* of E. In particular, the local frame of E^* on U associated to a frame $e_1, \cdots, e_r$ of $E_{|U}$ is the family of sections $e_1^*, \cdots, e_r^*$ of E^* on U characterized by the relations

$$\left\langle e_j^*, e_i \right\rangle = \delta_{ji}.$$

Remark 3.2.15. One checks easily that the canonical morphisms

$$E^* \otimes F \to \operatorname{Hom}(E, F)$$

is an isomorphism. So, if we know the Stiefel-Whitney classes of E^* and F, we can compute that of $\operatorname{Hom}(E, F)$.

Proposition 3.2.16. *For any real vector bundle E with base B and rank r, there is a (non-canonical) isomorphism*

$$E^* \simeq E.$$

In particular,

$$w(E^*) = w(E).$$

Proof. Using a partition of unity, one can construct easily on each fiber E_x an euclidean scalar product $(\cdot, \cdot)_b$ in such a way that

$$b \mapsto (e_j, e_k)_b$$

is a continuous function on U if $(e_1, \cdots, e_r)$ is a local frame of E on U. Using these scalar products, we obtain the requested isomorphism

$$\varphi : E \to E^*$$

by setting $\varphi_b(e) = (e, \cdot)_b$ for any $b \in B$ and any $e \in E_b$. □

Exercise 3.2.17.

(a) Show that

$$T\mathbb{P}_n(\mathbb{R}) \simeq \operatorname{Hom}(\mathbb{U}_n(\mathbb{R}), \mathbb{U}_n(\mathbb{R})^\perp)$$

where $\mathbb{U}_n(\mathbb{R})_b^\perp$ is the orthogonal complement of $\mathbb{U}_n(\mathbb{R})_b$ in $\mathbb{R}^{n+1}$ for any $b \in \mathbb{P}_n(\mathbb{R})$.

(b) Deduce from (a) that

$$w(T\mathbb{P}_n(\mathbb{R})) = \left(1 + e^{\mathbb{Z}_2}_{\mathbb{U}_n(\mathbb{R})}\right)^{n+1}.$$

(c) Prove that

$$w(T\mathbb{P}_n(\mathbb{R})) = 1$$

if and only if $n = 2^r - 1$ $(r \in \mathbb{N}_0)$. As a consequence, show that if $n+1$ is not a power of 2 then $\mathbb{P}_n(\mathbb{R})$ is not parallelizable.

Solution. (a) Let $d \in \mathbb{P}_n(\mathbb{R})$. By definition, d is a line of $\mathbb{R}^{n+1}$ containing the origin. Denote

$$q : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{P}_n(\mathbb{R})$$

the canonical map and let $v \in \mathbb{R}^{n+1} \setminus \{0\}$ be such that $q(v) = d$. Clearly, the linear map

$$T_v q : \mathbb{R}^{n+1} \to T_d\mathbb{P}_n(\mathbb{R})$$

has d as kernel. Hence,

$$T_v q_{|d^\perp} : d^\perp \to T_d\mathbb{P}_n(\mathbb{R})$$

is an isomorphism. Moreover, since

$$\begin{aligned}\operatorname{Hom}(\mathbb{U}_n(\mathbb{R}), \mathbb{U}_n(\mathbb{R})^\perp)_d &= \operatorname{Hom}(\mathbb{U}_n(\mathbb{R})_d, \mathbb{U}_n(\mathbb{R})_d^\perp)\\ &\simeq \operatorname{Hom}(d, d^\perp)\end{aligned}$$

v induces a canonical isomorphism

$$\epsilon_v : \operatorname{Hom}(\mathbb{U}_n(\mathbb{R}), \mathbb{U}_n(\mathbb{R})^\perp)_d \to d^\perp.$$

This isomorphism sends $h : d \to d^\perp$ to $h(v)$. By composition, we get the isomorphism

$$T_v q \circ \epsilon_v : \operatorname{Hom}(\mathbb{U}_n(\mathbb{R}), \mathbb{U}_n(\mathbb{R})^\perp)_d \to T_d\mathbb{P}_n(\mathbb{R}). \tag{*}$$

Since

$$q(\lambda v) = q(v)$$

for $\lambda \neq 0$, the chain rule shows that

$$T^q_{\lambda v}(\lambda \cdot \theta) = T^q_v(\theta)$$

for any $\theta \in \mathbb{R}^{n+1}$. Therefore, (*) does not depend on v and we get a canonical isomorphism

$$\mathrm{Hom}\,(\mathbb{U}_n(\mathbb{R}), \mathbb{U}_n(\mathbb{R})^\perp)_d \simeq T_d(\mathbb{P}_n(\mathbb{R})).$$

One checks easily that it extends to the requested isomorphism of vector bundles.

(b) Since

$$\mathbb{U}_n(\mathbb{R}) \oplus \mathbb{U}_n(\mathbb{R})^\perp = \mathbb{R}^{n+1} \times \mathbb{P}_n(\mathbb{R})$$

we see that

$$\begin{aligned}&\mathrm{Hom}\,(\mathbb{U}_n(\mathbb{R}), \mathbb{U}_n(\mathbb{R})) \oplus \mathrm{Hom}\,(\mathbb{U}_n(\mathbb{R}), \mathbb{U}_n(\mathbb{R})^\perp)\\ &\quad\simeq \mathrm{Hom}\,(\mathbb{U}_n(\mathbb{R}), \mathbb{R}^{n+1} \times \mathbb{P}_n(\mathbb{R}))\end{aligned}$$

and therefore that

$$(\mathbb{R} \times \mathbb{P}_n(\mathbb{R})) \oplus T\mathbb{P}_n(\mathbb{R}) \simeq [\mathbb{U}_n(\mathbb{R})^*]^{\oplus n+1}.$$

It follows that

$$1 \smile w(T\mathbb{P}_n(\mathbb{R})) = w(\mathbb{U}_n(\mathbb{R}))^{n+1},$$

hence the conclusion.

(c) Assume $n = 2^r - 1$ with $r \in \mathbb{N}$. It follows that

$$w(T\mathbb{P}_n(\mathbb{R})) = \left(1 + e^{\mathbb{Z}_2}_{\mathbb{U}_n(\mathbb{R})}\right)^{2^r} = 1 + \left[e^{\mathbb{Z}_2}_{\mathbb{U}_n(\mathbb{R})}\right]^{2^r}$$

since we work modulo 2. But $2^r = n + 1 > \dim \mathbb{P}_n(\mathbb{R})$ and we get

$$w(T\mathbb{P}_n(\mathbb{R})) = 1.$$

Assume now n is not of the preceding type. Then, $n = 2^r m - 1$ where m is an odd number which is strictly greater than 1. In this case $2^r \leq n$ and

$$\begin{aligned}w(T\mathbb{P}_n(\mathbb{R})) &= \left(1 + \left(e^{\mathbb{Z}_2}_{\mathbb{U}_n(\mathbb{R})}\right)^{2^r}\right)^m\\ &= 1 + m\left(e^{\mathbb{Z}_2}_{\mathbb{U}_n(\mathbb{R})}\right)^{2^r} + \cdots\end{aligned}$$

Since $m \not\equiv 0 \pmod 2$ and $2^r \leq \dim \mathbb{P}_n(\mathbb{R})$, it follows that $w(T\mathbb{P}_n(\mathbb{R})) \neq 1$. To conclude, it remains to note that if $\mathbb{P}_n(\mathbb{R})$ is parallelizable then $T\mathbb{P}_n(\mathbb{R})$ is trivializable and we have $w(T\mathbb{P}_n(\mathbb{R})) = 1$. $\square$

Exercise 3.2.18.

(a) Show that if a manifold M of dimension m can be immersed in $\mathbb{R}^{m+s}$ then the components of degree $k > s$ of $w^{-1}(TM)$ vanish.

(b) Deduce from (a) that $\mathbb{P}_{2^r}(\mathbb{R})$ cannot be immersed in $\mathbb{R}^{2^r+s}$ with $s < 2^r - 1$.

Solution. (a) Let $i : M \to \mathbb{R}^{m+s}$ be an immersion. Consider the associated exact sequence of bundles

$$0 \to TM \to i^{-1}T\mathbb{R}^{m+s} \to T_M\mathbb{R}^{m+s} \to 0.$$

Clearly,

$$1 = w(i^{-1}T\mathbb{R}^{m+1}) = w(TM)w(T_M\mathbb{R}^{m+s}).$$

It follows that $w(T_M\mathbb{R}^{m+s}) = w^{-1}(TM)$ and since $T_M\mathbb{R}^{m+s}$ has rank s, the conclusion follows.

(b) For $n = 2^r$, setting $e = e^{\mathbb{Z}_2}_{\mathbb{U}_n(\mathbb{R})}$, we have

$$w(T\mathbb{P}_n(\mathbb{R})) = (1 + e)^{2^r+1}.$$

Therefore,

$$w^{-1}(T\mathbb{P}_n(\mathbb{R})) = (1 + e + e^2 + \cdots + e^n)^{2^r+1}.$$

Using the fact that $(a + b)^2 = a^2 + b^2$ in $\mathrm{H}^{\cdot}(\mathbb{P}_n(\mathbb{R}); \mathbb{Z}_2)$, we get

$$\begin{aligned} w^{-1}(T\mathbb{P}_n(\mathbb{R})) &= (1 + e + e^2 + \cdots + e^n)(1 + e^{2^r}) \\ &= 1 + e + e^2 + \cdots + e^{n-1} \end{aligned}$$

and the conclusion follows. □

3.3 Homotopical classification of real vector bundles

Definition 3.3.1. Hereafter, we denote $G_{n,r}$ the Grassmannian formed by real vector subspaces of dimension r of $\mathbb{R}^n$.

Proposition 3.3.2. *The Grassmannian $G_{n,r}$ has a canonical structure of compact topological manifold of dimension $r(n - r)$.*

Proof. Denote $V_{n,r}$ the subset of $(\mathbb{R}^n)^r$ formed by the sequences $(v_1, \cdots, v_r)$ which are linearly independent. Since $(v_1, \cdots, v_r) \in V_{n,r}$ if and only if there is $i_1, \cdots i_r \in \{1, \cdots, n\}$ such that

$$\begin{vmatrix} v_{1i_1} & \cdots & v_{ri_1} \\ \vdots & \ddots & \vdots \\ v_{1i_r} & \cdots & v_{ri_r} \end{vmatrix} \neq 0$$

it is clear that $V_{n,r}$ is an open subset of $(\mathbb{R}^n)^r$. Clearly, $\mathrm{GL}_r(\mathbb{R})$ acts freely and continuously on the right on $V_{n,r}$. Denote $q : V_{n,r} \to G_{n,r}$ the map which sends a sequence $(v_1, \cdots, v_r)$ to its linear envelope and endow $G_{n,r}$ with the associated quotient topology. By construction,

$$q(v'_1, \cdots, v'_r) = q(v_1, \cdots, v_r)$$

if and only if there is $M \in \mathrm{GL}_r(\mathbb{R})$ such that

$$(v'_1, \cdots, v'_r) = (v_1, \cdots, v_r) \cdot M$$

or equivalently if and only if

$$rk(v'_1, \cdots, v'_r, v_1, \cdots, v_r) \leq r. \tag{*}$$

It follows that q identifies $V_{n,r}/\mathrm{GL}_r(\mathbb{R})$ with $G_{n,r}$. In particular, q is open and since condition (*) is clearly closed we see that $G_{n,r}$ is separated. Let L_0 be an element of $G_{n,r}$ and let L_1 be a supplement of L_0 in $\mathbb{R}^n$. Denote $p : \mathbb{R}^n \to L_0$ the projection associated to the decomposition $\mathbb{R}^n = L_0 \oplus L_1$. Set $U = \{L \in G_{n,r} : p(L) = L_0\}$. Clearly, U is an open subset of $G_{n,r}$. Moreover, since $p_{|L} : L \to L_0$ is bijective, any $L \in U$ may be viewed as the graph of a linear map $h : L_0 \to L_1$. This gives us a bijection

$$U \to \mathrm{Hom}\,(L_0, L_1)$$

which is easily checked to be an homeomorphism. Since $\mathrm{Hom}\,(L_0, L_1) \simeq \mathbb{R}^{r(n-r)}$, it follows that $G_{n,r}$ is a topological manifold of dimension $r(n-r)$. To prove that it is compact, remark that $G_{n,r} = q(S_{n,r})$ where $S_{n,r}$ is the compact subset of $V_{n,r}$ formed by orthonormed sequences. $\square$

Remark 3.3.3. The reader will easily complete the preceding proposition to show that $G_{n,r}$ has in fact a canonical structure of differential manifold.

Definition 3.3.4. We define $U_{n,r}$ as the subset of $\mathbb{R}^n \times G_{n,r}$ formed by the pairs (V, L) with $V \in L$.

Proposition 3.3.5. *The canonical projection $U_{n,r} \to G_{n,r}$ is a real vector bundle of rank r.*

Proof. The only non obvious part is to show that $U_{n,r}$ has locally a continuous frame. Using the notations introduced in the proof of the preceding proposition, we construct such a frame by choosing a basis $v_1, \cdots, v_r$ of L_0 and associating to any $L \in U$ the basis $p_{|L}^{-1}(v_1), \cdots, p_{|L}^{-1}(v_r)$. $\square$

Proposition 3.3.6. *Let B be a compact topological space. Assume E is a vector bundle of rank r on X. Then, there is $n \in \mathbb{N}$ and a continuous map*

$$f : B \to G_{n,r}$$

such that $f^{-1}(U_{n,r}) \simeq E$.

Proof. It is sufficient to construct a commutative diagram of continuous maps of the type

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "p_E"'] & U_{n,r} \arrow[r] \arrow[d, "p_{U_{n,r}}"] & \mathbb{R}^n \times G_{n,r} \arrow[dl] \\ B \arrow[r, "f"'] & G_{n,r} & \end{tikzcd}$$

with $g_{|E_b}$ injective. Note that in such a diagram f is determined by g. Moreover, if we denote q_1, q_2 the two projections of $\mathbb{R}^n \times G_{n,r}$ on $\mathbb{R}^n$ and $G_{n,r}$, we have $(q_2 \circ g)(e) = (q_1 \circ g)(E_{p_E(e)})$. Therefore, we have only to construct the continuous map $\widehat{f} := q_1 \circ g : E \to \mathbb{R}^n$ in such a way that $\widehat{f}_{|E_b}$ is injective and linear. To construct this map, let us proceed as follows. We cover B by a finite number of open subsets $U_1, \cdots, U_N$ on which E has a continuous frame. For each $k \in \{1, \cdots, N\}$ we choose a trivialization

$$\varphi_{U_k} : E_{|U_k} \xrightarrow{\sim} \mathbb{R}^r \times U_k$$

and set $h_{U_k} = q_1 \circ \varphi_{U_k}$. Denoting $(\psi_{U_1}, \cdots, \psi_{U_N})$ a partition of unity subordinated to the covering $\{U_1, \cdots, U_N\}$, we set

$$\widehat{f}(e) = ((\psi_{U_1} \circ p)(e) h_{U_1}(e), \cdots, (\psi_{U_N} \circ p)(e) h_{U_N}(e)) \in \mathbb{R}^{rN}.$$

It is clear that $\widehat{f}$ is continuous and linear on the fibers of E. It is also injective on the fibers of E. As a matter of fact, if e_1, $e_2 \in E_b$ are such that $\widehat{f}(e_1) = \widehat{f}(e_2)$, there is $k \in \{1 \cdots, N\}$ such that $(\psi_{U_k} \circ p)(b) \neq 0$. For such a k, we get

$$h_{U_k}(e_1) = h_{U_k}(e_2)$$

and hence $e_1 = e_2$. □

Definition 3.3.7. We denote $\mathbb{R}^\infty$ the real vector space formed by the sequences

$$(x_n)_{n \in \mathbb{N}}$$

of real numbers for which $\{n \in \mathbb{N} : x_n \neq 0\}$ is finite and endow it with the topology of the inductive limit

$$\varinjlim_{n \in \mathbb{N}} \mathbb{R}^n$$

corresponding to the transition morphisms

$$t_{m,n} : \mathbb{R}^n \to \mathbb{R}^m$$

being defined by setting

$$t_{m,n}(x) = (x, 0).$$

We denote $V_{\infty,r}$ the topological subspace of $(\mathbb{R}^\infty)^r$ formed by the sequences

$$(v_1, \cdots, v_r)$$

which are linearly independent. We denote $G_{\infty,r}$ the set of r-dimensional vector subspace of $\mathbb{R}^\infty$ and endow it with the quotient topology associated with the map

$$q : V_{\infty,r} \to G_{\infty,r}$$

which sends a sequence $(v_1, \cdots, v_r)$ to its linear envelope. Finally, we denote $U_{\infty,r}$ the subset of $\mathbb{R}^\infty \times G_{\infty,r}$ formed by the pairs (v, L) with $v \in L$.

Lemma 3.3.8. *Let $(X_n, x_{m,n})_{n\in\mathbb{N}}$ be an inductive system of locally compact topological spaces. Assume that Y_n is a closed subspace of X_n and that*

$$x^{-1}_{m,n}(Y_m) = Y_n$$

and denote $y_{m,n}$ the map $x_{m,n|Y_n} : Y_n \to Y_m$. Then, $(Y_n, y_{m,n})_{n\in\mathbb{N}}$ is an inductive system and $\varinjlim_{n\in\mathbb{N}} Y_n$ is a closed subspace of $\varinjlim_{n\in\mathbb{N}} X_n$.

Proof. Set $X = \varinjlim_{n\in\mathbb{N}} X_n$ and $Y = \varinjlim_{n\in\mathbb{N}} Y_n$ and identify Y with a subset of X. Denote $x_n : X_n \to X$, $y_n : Y_n \to Y$ the canonical maps.

(a) Let $v \in Y$ and let V be an open neighborhood of v in Y. We have to show that there is a neighborhood U of v in X such that $U \cap Y \subset V$. We know that $v = y_n(v_n)$ for some $n \in \mathbb{N}$ and that $V_n := y_n^{-1}(V)$ is a neighborhood of v_n in Y_n. Let K_n be a compact neighborhood of v_n in X_n such that $K_n \cap Y_n \subset V_n$ and let us construct by induction a sequence $(K_m)_{m\geq n}$ such that

(i) K_m is a compact neighborhood of $v_m := y_{m,n}(v_n)$ in X_m,

(ii) $K_m \cap Y_m \subset V_m := y_m^{-1}(V)$,

(iii) $x_{m+1,m}(K_m) \subset K^\circ_{m+1}$.

This is possible since if K_m is a compact neighborhood of v_m in X_m such that $K_m \cap Y_m \subset V_m$, we can construct K_{m+1} as follows. Using the fact that $x_{m+1,m}(K_m) \cap Y_{m+1}$ is a compact subset of X_{m+1} included in V_{m+1} together with the fact that any $w_{m+1} \in V_{m+1}$ has a compact neighborhood W_{m+1} with $W_{m+1} \cap Y_{m+1} \subset V_{m+1}$, it is easy to obtain a compact subset L_{m+1} of X_{m+1} such that $L_{m+1} \cap Y_{m+1} \subset V_{m+1}$, $L^\circ_{m+1} \supset x_{m+1,m}(K_m) \cap Y_{m+1}$. Since $x_{m+1,m}(K_m) \setminus L^\circ_{m+1}$ is a compact subset of X_{m+1} disjoint from Y_{m+1}, it has a compact neighborhood L'_{m+1} disjoint from Y_{m+1}. Taking $K_{m+1} = L_{m+1} \cup L'_{m+1}$, we see that $K_{m+1} \cap Y_{m+1} \subset V_{m+1}$ and that $K^\circ_{m+1} \supset x_{m+1,m}(K_m)$ as requested. Now, set $K = \varinjlim_{m \geq n} K_m$. By construction, $K \cap Y \subset V$. Moreover, since $x_{m+1,m}(K_m) \subset K^\circ_{m+1}$, K is an open neighborhood of v in X.

(b) Let $u \in X \setminus Y$. We know that $u = x_n(u_n)$ for some $n \in \mathbb{N}$ and that for any $m \geq n$, $u_m := x_{m,n}(u_n) \notin Y_m$. Working as in (a), it is clearly possible to construct by induction a sequence $(K_m)_{m \geq n}$ such that

(i) K_m is a compact neighborhood of u_m in X_m,

(ii) $K_m \cap Y_m = \emptyset$,

(iii) $x_{m+1,m}(K_m) \subset K^\circ_{m+1}$.

Set $K = \varinjlim_{m \geq n} K_m$. By construction, $K \cap Y = \emptyset$ and K is an open neighborhood of u; hence the conclusion. □

Lemma 3.3.9. *Let $(X_n, x_{m,n})_{n \in \mathbb{N}}$ and $(Y_n, y_{m,n})_{n \in \mathbb{N}}$ be inductive systems of locally compact topological spaces. Then,*

$$\varinjlim_{n \in \mathbb{N}} (X_n \times Y_n) \simeq (\varinjlim_{n \in \mathbb{N}} X_n) \times (\varinjlim_{n \in \mathbb{N}} Y_n)$$

as topological spaces.

Proof. Set $Z_n = X_n \times Y_n$ and set

$$X = \varinjlim_{n \in \mathbb{N}} X_n, \quad Y = \varinjlim_{n \in \mathbb{N}} Y_n, \quad Z = \varinjlim_{n \in \mathbb{N}} Z_n.$$

Denote $x_n : X_n \to X$, $y_n : Y_n \to Y$, $z_n : Z_n \to Z$ the canonical maps. Since the canonical continuous map

$$i : Z \to X \times Y$$

is clearly bijective, it remains to show that if W is a neighborhood of w in Z and $i(w) = (u, v)$, then there are neighborhoods U, V of u, v in X and Y such that $i(W) \supset U \times V$. We know that $w = z_n(w_n)$ for some $n \in \mathbb{N}$ and some $w_n \in Z_n$. Set $w_m = z_{m,n}(w_n)$ and $W_m = z_m^{-1}(W)$ for any $m \geq n$. Let $u_n \in X_n$, $v_n \in Y_n$ be such that $i(w_n) = (u_n, v_n)$. Since $i(W_n)$ is a neighborhood of (u_n, v_n) in $X_n \times Y_n$, there are compact neighborhoods K_n, L_n of u_n, v_n in X_n and Y_n such that $i(W_n) \supset K_n \times L_n$. Starting with these neighborhoods, let us construct by induction sequences $(K_m)_{m \geq n}$, $(L_m)_{m \geq n}$ such that

(i) K_m (resp. L_m) is a compact neighborhood of u_m (resp. v_m),

(ii) $K_m \times L_m \subset i(W_m)$,

(iii) $x_{m+1,m}(K_m) \subset K^{\circ}_{m+1}$, $y_{m+1,m}(L_m) \subset L^{\circ}_{m+1}$.

This is possible since $i(W_{n+1})$ is a neighborhood of

$$x_{m+1,m}(K_m) \times y_{m+1,m}(L_m).$$

Set

$$K = \varinjlim_{m \geq n} K_m, \quad L = \varinjlim_{m \geq n} L_m.$$

By construction, K and L are open subsets of X and Y and $i(W) \supset K \times L$. The conclusion follows. $\square$

Proposition 3.3.10. *We have*

$$V_{\infty,r} \simeq \varinjlim_{n \geq r} V_{n,r},$$

$$G_{\infty,r} \simeq \varinjlim_{n \geq r} G_{n,r},$$

$$U_{\infty,r} \simeq \varinjlim_{n \geq r} U_{n,r}.$$

In particular, $V_{\infty,r}$ is an open subset of $\mathbb{R}^\infty$ and the canonical projection

$$U_{\infty,r} \to G_{\infty,r}$$

is a real vector bundle of rank r.

Proof. Thanks to Lemma 3.3.9, $(\mathbb{R}^\infty)^r \simeq \varinjlim_{n \in \mathbb{N}} (\mathbb{R}^n)^r$ and it follows from the fact $V_{\infty,r} \cap (\mathbb{R}^n)^r = V_{n,r}$ that $V_{\infty,r}$ is open in $(\mathbb{R}^\infty)^r$ and that

$$V_{\infty,r} \simeq \varinjlim_{n \geq r} V_{n,r}.$$

The relation

$$G_{\infty,r} \simeq \varinjlim_{n \geq r} G_{n,r}$$

follows easily. Since $U_{n,r}$ is a closed subspace of $\mathbb{R}^n \times G_{n,r}$, the relation

$$U_{\infty,r} = \varinjlim_{n \geq r} U_{n,r}$$

follows directly from Lemma 3.3.8 together with the relation

$$\mathbb{R}^\infty \times G_{\infty,r} = \varinjlim_{n \geq r} \mathbb{R}^n \times G_{n,r}$$

which follows from Lemma 3.3.9. □

Remark 3.3.11. Although we will not prove it here, the space $G_{\infty,r}$ is paracompact (see e.g. [20]).

Lemma 3.3.12. *Let B be a topological space and let E be a real vector bundle of rank r on B. Then, there is a locally finite countable family $(V_k)_{k\in\mathbb{N}}$ of open subsets of B such that*

(a) $\bigcup_{k\in\mathbb{N}} V_k = B$;

(b) $E_{|V_k} \simeq \mathbb{R}^r \times V_k$.

Proof. Let $\mathcal{U}$ be a locally finite covering of B by open subsets U on which E is trivializable and let $(\varphi_U)_{U\in\mathcal{U}}$ be continuous partition of unity subordinated to $\mathcal{U}$. For any non-empty finite subset $\mathcal{S}$ of $\mathcal{U}$, set

$$V(\mathcal{S}) = \{b \in B | \inf_{U\in\mathcal{S}} \varphi_U(b) > \sup_{U\in\mathcal{U}\setminus\mathcal{S}} \varphi_U(b).\}$$

Set also

$$V_k = \bigcup_{\#\mathcal{S}=k} V(\mathcal{S}).$$

One checks easily that $V(\mathcal{S})$ is an open subset of $\bigcap \mathcal{S}$. It follows that E is trivializable on $V(\mathcal{S})$ and that $(V_k)_{k\in\mathbb{N}}$ is a locally finite family. Since

$$V(\mathcal{S}) \cap V(\mathcal{S}') \neq \emptyset$$

entails that $\mathcal{S} \subset \mathcal{S}'$ or $\mathcal{S}' \subset \mathcal{S}$, we see that $V(\mathcal{S})$ and $V(\mathcal{S}')$ are disjoint if $\mathcal{S}$ and $\mathcal{S}'$ are distinct finite subsets of $\mathcal{U}$ with $k > 0$ elements. It follows that E is trivializable on V_k. To conclude, note that we have

$$\bigcup_{k\in\mathbb{N}} V_k = B$$

since any $b \in B$ belongs to $V(\mathcal{S})$ with

$$\mathcal{S} = \{U \in \mathcal{U} : \varphi_U(b) > 0\}.$$

□

Theorem 3.3.13. *For any topological space B and any real vector bundle E of rank r on B, there is a continuous map*

$$f : B \to G_{\infty,r}$$

such that $f^{-1}(U_{\infty,r}) \simeq E$. Moreover, such a map is unique up to homotopy.

Proof. Thanks to the preceding Lemma, the first part is obtained by working as in the proof of Proposition 3.3.6.

For the second part, we have to show that given two commutative diagrams of the type

$$\begin{tikzcd} E \arrow[r, "g_0"] \arrow[d] & U_{\infty,r} \arrow[r] \arrow[d] & \mathbb{R}^\infty \times G_{\infty,r} \arrow[dl] \\ B \arrow[r, "f_0"'] & G_{\infty,r} & \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "g_1"] \arrow[d] & U_{\infty,r} \arrow[r] \arrow[d] & \mathbb{R}^\infty \times G_{\infty,r} \arrow[dl] \\ B \arrow[r, "f_1"'] & G_{\infty,r} & \end{tikzcd}$$

where g_0 and g_1 are injective on the fibers of E, there is a homotopy between f_0 and f_1. Setting $\widehat{f_0} = q_1 \circ g_0$ and $\widehat{f_1} = q_1 \circ g_1$, we get two maps

$$\widehat{f_0} : E \to \mathbb{R}^\infty, \qquad \widehat{f_1} : E \to \mathbb{R}^\infty$$

which are linear and injective on the fibers of E and whose knowledge makes it possible to reconstruct the whole diagrams.

(a) Assume $\widehat{f_0}(e) \notin \{\lambda \widehat{f_1}(e) : \lambda < 0\}$ for $e \in \dot{E}$. Then, for $t \in [0,1]$, the continuous map $\widehat{h}_t : E \to \mathbb{R}^\infty$ defined by setting

$$\widehat{f_t}(e) = (1-t)\widehat{f_0}(e) + t\widehat{f_1}(e)$$

is linear and injective on the fibers of E. Hence it gives rise to a commutative diagram

$$\begin{tikzcd} E \arrow[r, "g_t"] \arrow[d] & U_{\infty,r} \arrow[r] \arrow[d] & \mathbb{R}^\infty \times G_{\infty,r} \arrow[dl] \\ B \arrow[r, "f_t"'] & G_{\infty,r} & \end{tikzcd}$$

here $q_1 \circ g_t = \widehat{h}_t$ and g_t is injective on the fibers of E. The conclusion follows since f_t $(t \in [0,1])$ is a homotopy between f_0 and f_1.

(b) In general, consider the maps

$$\alpha : \mathbb{R}^\infty \to \mathbb{R}^\infty, \qquad \beta : \mathbb{R}^\infty \to \mathbb{R}^\infty$$

defined by setting

$$\alpha(e_i) = e_{2i}, \qquad \beta(e_i) = e_{2i+1}$$

where e_i are the vectors of the canonical basis of $\mathbb{R}^\infty$. Set $\widehat{f}_\alpha = \alpha \circ \widehat{f}_0$ and $\widehat{f}_\beta = \beta \circ \widehat{f}_1$ and consider the associated diagrams

$$\begin{tikzcd} E \arrow[r, "g_\alpha"] \arrow[d] & U_{\infty,r} \arrow[r] \arrow[d] & \mathbb{R}^\infty \times G_{\infty,r} \arrow[dl] \\ B \arrow[r, "f_\alpha"'] & G_{\infty,r} & \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "g_\beta"] \arrow[d] & U_{\infty,r} \arrow[r] \arrow[d] & \mathbb{R}^\infty \times G_{\infty,r} \arrow[dl] \\ B \arrow[r, "f_\beta"'] & G_{\infty,r} & \end{tikzcd}$$

here $q_1 \circ g_\alpha = \widehat{g}_\alpha$ and $q_1 \circ g_\beta = \widehat{g}_\beta$. By (a), we have $f_0 \sim f_\alpha$, $f_\alpha \sim f_\beta$, $f_\beta \sim f_1$; hence the conclusion. $\square$

Corollary 3.3.14. *Assume B is a topological space. Then, the map*

$$[B, G_{\infty,r}] \to \mathrm{Isom}(\mathcal{V}\mathrm{ect}^r_{\mathbb{R}}(B))$$

which associates to the homotopy class of $f : B \to G_{\infty,r}$ the isomorphism class of $f^{-1}(U_{\infty,r})$ is a bijection.

Proof. Since Lemma 3.3.15 shows that the map considered above is well-defined, the conclusion follows easily from the preceding result. $\square$

Lemma 3.3.15. *Assume B is a topological space. Then, for any real vector bundle F on $B \times [0, 1]$, there is a real vector vector bundle E on B and a non-canonical isomorphism*

$$F \simeq p_B^{-1} E.$$

In particular, if $h : B \times I \to B'$ is a continuous homotopy and E' is a real vector bundle on B', then the isomorphy class of

$$h_t^{-1} E'$$

does not depend on $t \in [0, 1]$.

Proof. See e.g. [16, p. 28]. $\square$

3.4 Characteristic classes

Definition 3.4.1. A *characteristic class* with coefficients in the abelian group M for real vector bundles of rank r is a law γ which associates to any real vector bundle E of rank r and base X a class $\gamma(E) \in \mathrm{H}^{\cdot}(X; A)$ in such a way that

$$\gamma(f^{-1}(E)) = f^* \gamma(E)$$

for any continuous map $f : Y \to X$.

Proposition 3.4.2. *Characteristic classes with coefficients in M for real vector bundles of rank r are canonically in bijection with*

$$\mathrm{H}^{\cdot}(G_{\infty,r}; M).$$

Proof. Thanks to Theorem 3.3.13, we know that any real vector bundle E of rank r on B may be written as

$$f^{-1}(U_{\infty,r})$$

where $f : B \to G_{\infty,r}$ is uniquely determined up to homotopy by E. If γ is a characteristic class, it follows that

$$\gamma(E) = f^*\gamma(U_{\infty,r})$$

and γ is uniquely determined by $\gamma(U_{\infty,r}) \in \mathrm{H}^{\cdot}(G_{\infty,r}; M)$. Moreover, if

$$c \in \mathrm{H}^{\cdot}(G_{\infty,r}; M)$$

is given, we can construct a characteristic class γ such that $\gamma(U_{\infty,r}) = c$ by setting

$$\gamma(E) = f^* c$$

since f^* depends only on E. □

Lemma 3.4.3. *Let X be a topological space. Assume $(A_i)_{i\in I}$ is a directed family of subspaces of X such that*

$$X \simeq \varinjlim_{i\in I} A_i.$$

Then, for any sheaf $\mathcal{F}$ on X, we have

$$\Gamma(X; \mathcal{F}) \simeq \varprojlim_{i\in I} \Gamma(A_i; \mathcal{F}).$$

Proof. The fact that the canonical map

$$\Gamma(X; \mathcal{F}) \to \varprojlim_{i\in I} \Gamma(A_i; \mathcal{F})$$

is injective is a direct consequence of the fact that $X = \bigcup_{i\in I} A_i$. Let us prove that it is also surjective. Let $(\sigma_i)_{i\in I}$ be an element of $\varprojlim_{i\in I} \Gamma(A_i; \mathcal{F})$. Define the family $(s_x)_{x\in X}$ of $\prod_{x\in X} \mathcal{F}_x$ by setting

$$s_x = (\sigma_i)_x$$

for any $x \in A_i$. We have to prove that $(s_x)_{x\in X}$ comes locally from a section of $\mathcal{F}$. Fix $x \in X$. There is $i \in I$ such that $x \in A_i$, an open neighborhood U of x in X and $\sigma \in \mathcal{F}(U)$ such that $\sigma_{|A_i\cap U} = \sigma_{i|A_i\cap U}$. Set $V = \{x \in U : \sigma_x = s_x\}$. Clearly,

$$V \cap A_j = \{x \in U \cap A_j : \sigma_x = (\sigma_j)_x\}$$

is open in A_j. Therefore, V is open in X and the conclusion follows. ☐

Lemma 3.4.4. *Let X be a topological space and let $(F_n)_{n\in\mathbb{N}}$ be an increasing sequence of closed subspaces of X such that*

$$X \simeq \varinjlim_{n\in\mathbb{N}} F_n.$$

Then, for any sheaf $\mathcal{F}$ on X, we have

$$\mathrm{R}\Gamma(X;\mathcal{F}) \simeq \mathrm{R}\varprojlim_{n\in\mathbb{N}} \mathrm{R}\Gamma(F_n;\mathcal{F}).$$

In particular, if

$$\mathrm{H}^k(F_n;\mathcal{F})$$

satisfies Mittag-Leffler condition for $k \leq l$, then

$$\mathrm{H}^l(X;\mathcal{F}) \simeq \varprojlim_{n\in\mathbb{N}} \mathrm{H}^l(F_n;\mathcal{F}).$$

Proof. It is sufficient to apply Lemma 3.4.3 to a soft resolution of $\mathcal{F}$ and to use the fact that Mittag-Leffler condition implies $\varprojlim$ acyclicity. ☐

Proposition 3.4.5. *For $n \geq r$, the $\mathbb{Z}_2$-algebra $\mathrm{H}^{\cdot}(G_{n,r};\mathbb{Z}_2)$ is generated by*

$$w_1(U_{n,r}), \cdots, w_r(U_{n,r}).$$

Proof. Recall that the universal bundle of rank r

$$p : U_{n,r} \to G_{n,r}$$

is defined by setting

$$U_{n,r} = \{(v, L) \in \mathbb{R}^n \times G_{n,r} : v \in L\}.$$

Define

$$U_{n,r}^{\perp} = \{(v, L) \in \mathbb{R}^n \times G_{n,r} : v \in L^{\perp}\}$$

and denote

$$q : U_{n,r}^{\perp} \to G_{n,r}$$

the canonical projection. One can check easily that q is a real vector bundle of rank $n-r$. Denote $\dot{U}_{n,r}$ (resp. $\dot{U}^{\perp}_{n,r}$) the space $U_{n,r}$ (resp. $U^{\perp}_{n,r}$) without its zero section and denote $\dot{p}$ (resp. $\dot{q}$) the canonical projection of $\dot{U}_{n,r}$ (resp.$\dot{U}^{\perp}_{n,r}$) on $G_{n,r}$. Define the map

$$f : \dot{U}_{n,r} \to \dot{U}^{\perp}_{n,r-1}$$

by setting $f(v,L) = (v, L \cap \rangle v \langle^{\perp})$ for $v \in \mathbb{R}^n \setminus \{0\}$. One checks directly that this application is a homeomorphism; the inverse being given by the map

$$f^{-1} : \dot{U}^{\perp}_{n,r-1} \to \dot{U}_{n,r}$$

defined by setting

$$f^{-1}(v, L') = (v, L' + \rangle v \langle)$$

for any $v \in L'^{\perp} \setminus \{0\}$. It follows that the Gysin sequences for $U_{n,r}$ and $U^{\perp}_{n,r-1}$:

$$\cdots \mathrm{H}^{k-r}(G_{n,r};\mathbb{Z}_2) \xrightarrow{\smile e^{\mathbb{Z}_2}_{U_{n,r}}} \mathrm{H}^k(G_{n,r};\mathbb{Z}_2) \xrightarrow{\dot{p}^*} \mathrm{H}^k(\dot{U}_{n,r};\mathbb{Z}_2) \to \cdots$$

$$\cdots \mathrm{H}^{k-n+r-1}(G_{n,r-1};\mathbb{Z}_2) \xrightarrow{\smile e^{\mathbb{Z}_2}_{U^{\perp}_{n,r-1}}} \mathrm{H}^k(G_{n,r-1};\mathbb{Z}_2) \xrightarrow{\dot{q}^*} \mathrm{H}^k(\dot{U}^{\perp}_{n,r-1};\mathbb{Z}_2) \to \cdots$$

are connected by the isomorphism

$$f^* : \mathrm{H}^k(\dot{U}^{\perp}_{n,r-1};\mathbb{Z}_2) \to \mathrm{H}^k(\dot{U}_{n,r};\mathbb{Z}_2).$$

Let us prove that

$$\dot{p}^*(w(U_{n,r})) = f^*(\dot{q}^*(w(U_{n,r-1}))). \tag{*}$$

To this end, consider the commutative diagram

$$\begin{array}{ccc} \dot{U}_{n,r} & \xrightarrow{\dot{q}\circ f} & G_{n,r-1} \\ \uparrow & & \uparrow \\ \dot{p}^{-1}U_{n,r} & \xrightarrow[g]{} & U_{n,r-1} \end{array}$$

where the map g is defined by setting

$$g(v', v, L) = (v'', L \cap \rangle v \langle^{\perp})$$

where $L \in G_{n,r}$, $v \in L \setminus \{0\}$, $v' \in L$ and v'' is the orthogonal projection of v' on $\rangle v \langle^{\perp}$. This gives us a morphism

$$\dot{p}^{-1}U_{n,r} \to (\dot{q} \circ f)^{-1}U_{n,r-1}$$

which is surjective. Its kernel is given by

$$L_{n,r} = \{(v', v, L) : (v, L) \in \dot{U}_{n,r}, v' \in \rangle v \langle\}$$

which is a trivializable line bundle on $\dot{U}_{n,r}$. From the exact sequence

$$0 \to L_{n,r} \to \dot{p}^{-1} U_{n,r} \to (\dot{q} \circ f)^{-1} U_{n,r-1} \to 0$$

we get that

$$w(\dot{p}^{-1} U_{n,r}) = w(L_{n,r}) \smile w((\dot{q} \circ f)^{-1} U_{n,r-1}) = w((\dot{q} \circ f)^{-1} U_{n,r-1})$$

and hence that

$$\dot{p}^* w(U_{n,r}) = f^*(\dot{q}^*(w(U_{n,r-1}))).$$

We will now prove by increasing induction on $k \geq 0$ and $r \geq 1$ that for $n \geq r$, any $c^k \in \mathrm{H}^k(G_{n,r}; \mathbb{Z}_2)$ may be expressed as a polynomial $R(w_1(U_{n,r}), \cdots, w_r(U_{n,r}))$ in the Stiefel-Whitney classes of $U_{n,r}$. The starting point will be the case $k = 0$, $r = 1$ which is obvious. From the two "exact triangles"

$$\begin{tikzcd}
\mathrm{H}^{\cdot}(G_{n,r}; \mathbb{Z}_2) \arrow[rr, "\smile e^{\mathbb{Z}_2}_{U_{n,r}}"] & & \mathrm{H}^{\cdot}(G_{n,r}; \mathbb{Z}_2) \arrow[dl, "\dot{p}^*"] \\
& \mathrm{H}^{\cdot}(\dot{U}_{n,r}; \mathbb{Z}_2) \arrow[ul] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{H}^{\cdot}(G_{n,r-1}; \mathbb{Z}_2) \arrow[rr, "\smile e^{\mathbb{Z}_2}_{U^{\perp}_{n,r-1}}"] & & \mathrm{H}^{\cdot}(G_{n,r-1}; \mathbb{Z}_2) \arrow[dl, "\dot{q}^*"] \\
& \mathrm{H}^{\cdot}(\dot{U}^{\perp}_{n,r-1}; \mathbb{Z}_2) \arrow[ul] &
\end{tikzcd}$$

and Lemma 3.4.6 below, we deduce that

$$\begin{aligned}
\dim \operatorname{Im} \dot{p}^* &= \frac{1}{2} \dim \mathrm{H}^{\cdot}(\dot{U}_{n,r}; \mathbb{Z}_2) \\
&= \frac{1}{2} \dim \mathrm{H}^{\cdot}(\dot{U}^{\perp}_{n,r-1}; \mathbb{Z}_2) \\
&= \dim \operatorname{Im} \dot{q}^* = \dim \operatorname{Im}(f^* \circ \dot{q}^*).
\end{aligned}$$

Moreover, it follows from (*) and the induction hypothesis that

$$\operatorname{Im} \dot{p}^* \supset \operatorname{Im}(f^* \circ \dot{q}^*).$$

Putting these two facts together, we see that $\operatorname{Im} \dot{p}^* = \operatorname{Im}(f^* \circ \dot{q}^*)$. Let us fix a class c in $\mathrm{H}^k(G_{n,r}; \mathbb{Z}_2)$. From what precedes, we know that there is $c' \in \mathrm{H}^k(G_{n,r-1}; \mathbb{Z}_2)$ such that

$$\dot{p}^*(c') = f^*(\dot{q}^*(c)).$$

By the induction hypothesis, there is a polynomial relation

$$c' = R'(w_1(U_{n,r-1}), \cdots, w_{r-1}(U_{n,r-1})).$$

Using (*), it follows that

$$\dot{p}^*(c) = \dot{p}^* R'(w_1(U_{n,r}), \cdots, w_{r-1}(U_{n,r}))$$

and hence that

$$c - R'(w_1(U_{n,r}), \cdots, w_{r-1}(U_{n,r})) = c'' \smile e^{\mathbb{Z}_2}_{U_{n,r}} = c'' \smile w_r(U_{n,r})$$

with $c'' \in \mathrm{H}^{k-2}(G_{n,r}; \mathbb{Z}_2)$. By the induction hypothesis,

$$c'' = R''(w_1(U_{n,r}), \cdots, w_r(U_{n,r}))$$

and the conclusion follows. □

Lemma 3.4.6. *Let*

$$\begin{tikzcd} E \arrow[rr, "u"] & & E \arrow[dl, "v"] \\ & F \arrow[ul, "w"] & \end{tikzcd}$$

be an exact triangle of vector spaces (i.e. $\operatorname{Ker} w = \operatorname{Im} v$, $\operatorname{Ker} v = \operatorname{Im} u$, $\operatorname{Ker} u = \operatorname{Im} w$*). Assume E has finite dimension. Then, F has finite dimension and*

$$\dim \operatorname{Im} v = \frac{1}{2} \dim F.$$

Proof. We know that

$$\operatorname{Im} v \simeq \operatorname{Coker} u, \qquad \operatorname{Coker} v \simeq \operatorname{Im} w \simeq \operatorname{Ker} u.$$

Since

$$\dim \operatorname{Im} u + \dim \operatorname{Ker} u = \dim E,$$

we have

$$\dim \operatorname{Coker} u = \dim \operatorname{Ker} u.$$

It follows that

$$\dim \operatorname{Im} v = \dim \operatorname{Coker} v < +\infty.$$

Hence,

$$\dim F = \dim \operatorname{Im} v + \dim \operatorname{Coker} v = 2 \dim \operatorname{Im} v.$$

□

Proposition 3.4.7. *The canonical morphism*

$$\mathbb{Z}_2[W_1, \cdots, W_r] \to \mathrm{H}^{\cdot}(G_{\infty,r}; \mathbb{Z}_2)$$

which sends $W_k \mapsto w_k(U_{\infty,r})$ *is an isomorphism.*

Proof. Using the notations introduced in the proof of Proposition 3.4.5, we get from the Gysin sequences that, for $k < n - r$, we have the exact sequences

$$0 \to \mathrm{H}^k(G_{n,r-1}; \mathbb{Z}_2) \xrightarrow{\dot{q}^*} \mathrm{H}^k(\dot{U}^{\perp}_{n,r-1}; \mathbb{Z}_2) \to 0$$

and

$$\cdots \mathrm{H}^{k-r}(G_{n,r}; \mathbb{Z}_2) \xrightarrow{\smile e^{\mathbb{Z}_2}_{U_{n,r}}} \mathrm{H}^k(G_{n,r}; \mathbb{Z}_2) \xrightarrow{\dot{p}^*} \mathrm{H}^k(\dot{U}_{n,r}; \mathbb{Z}_2) \to \cdots$$

It follows from the equality $\mathrm{Im}(f^* \circ \dot{q}^*) = \mathrm{Im}\,\dot{p}^*$ that $\dot{p}^*$ is surjective in degree $k < n - r$. Hence, we get the exact sequence

$$0 \to \mathrm{H}^{k-r}(G_{n,r}; \mathbb{Z}_2) \xrightarrow{\smile e^{\mathbb{Z}_2}_{U_{n,r}}} \mathrm{H}^k(G_{n,r}; \mathbb{Z}_2) \xrightarrow{\lambda} \mathrm{H}^k(G_{n,r-1}; \mathbb{Z}_2) \to 0 \qquad (*)$$

where $\lambda = (\dot{q}^*)^{-1} \circ f^{*\,-1} \circ \dot{p}^*$. Thanks to the formula (*) of the proof of Proposition 3.4.5 we have $\lambda(w(U_{n,r})) = w(U_{n,r-1})$. Since $U_{n,r|G_{n-1,r}} = U_{n-1,r}$, it follows from Proposition 3.4.5 that the restriction map

$$\mathrm{H}^k(G_{m,r}; \mathbb{Z}_2) \to \mathrm{H}^k(G_{n,r}; \mathbb{Z}_2)$$

is surjective for $m \geq n$. Hence, Lemma 3.4.4 shows that

$$\mathrm{H}^k(G_{\infty,r}; \mathbb{Z}_2) \simeq \varprojlim_{n \geq r} \mathrm{H}^k(G_{n,r}; \mathbb{Z}_2).$$

Taking the projective limit of the sequences (*) and using the fact that the Mittag-Leffler condition implies $\varprojlim$-acyclicity, we get the exact sequence

$$0 \to \mathrm{H}^{k-r}(G_{\infty,r}; \mathbb{Z}_2) \xrightarrow{\smile e^{\mathbb{Z}_2}_{U_{\infty,r}}} \mathrm{H}^k(G_{\infty,r}; \mathbb{Z}_2) \to \mathrm{H}^k(G_{\infty,r-1}; \mathbb{Z}_2) \to 0.$$

Working by induction, as in the proof of Proposition 3.4.5, we see easily that any $c \in \mathrm{H}^k(G_{\infty,r}; \mathbb{Z}_2)$ may be written polynomially as

$$c = R(w_1(U_{\infty,r}), \cdots, w_r(U_{\infty,r})).$$

To show the uniqueness of such a writing, let us proceed by induction on $k \geq 0$ and $r \geq 1$ as follows. Assume

$$R(w_1(U_{\infty,r}), \cdots, w_r(U_{\infty,r})) = S(w_1(U_{\infty,r}), \cdots, w_r(U_{\infty,r})). \qquad (*)$$

Write

$$R(w_1, \cdots, w_r) = R'(w_1, \cdots, w_{r-1}) + w_r R''(w_1, \cdots, w_r)$$

$$S(w_1, \cdots, w_r) = S'(w_1, \cdots, w_{r-1}) + w_r S''(w_1, \cdots, w_r).$$

Applying λ to (*), we see that

$$R'(w_1(U_{\infty,r-1}), \cdots, w_{r-1}(U_{\infty,r-1})) = S'(w_1(U_{\infty,r-1}), \cdots, w_{r-1}(U_{\infty,r-1})).$$

By the induction hypothesis, we get $R' = S'$ and, hence,

$$\begin{aligned} w_r(U_{\infty,r}) &\smile R''(w_1(U_{\infty,r}), \cdots, w_r(U_{\infty,r})) \\ &= w_r(U_{\infty,r}) \smile S''(w_1(U_{\infty,r}), \cdots, w_r(U_{\infty,r})). \end{aligned}$$

Since $w_r(U_{\infty,r}) = e^{\mathbb{Z}_2}_{U_{\infty,r}}$ and $e^{\mathbb{Z}_2}_{U_{\infty,r}} \smile \cdot$ is injective, the conclusion follows by the induction hypothesis. ☐

3.5 Cohomological classification of real vector bundles

Definition 3.5.1. Let G be a topological group and let $\mathcal{U}$ be an open covering of the topological space X. A continuous Čech 1-cochain on $\mathcal{U}$ with values in G is the datum for any $U, V \in \mathcal{U}$ of a continuous map

$$\psi_{VU} : U \cap V \to G.$$

Such a cochain is a cocycle if

$$\psi_{WV} \circ \psi_{VU} = \psi_{WU} \qquad \text{on } U \cap V \cap W.$$

Two continuous 1-cocycles $(\psi_{UV})_{V,U \in \mathcal{U}}$, $(\psi'_{VU})_{V,U \in \mathcal{U}}$ are equivalent if we can find for any $U \in \mathcal{U}$ a continuous map

$$\psi_U : U \to G$$

in such a way that

$$\psi'_{VU} = \psi_V \circ \psi_{VU} \circ \psi_U^{-1} \qquad \text{on } U \cap V.$$

The set of equivalence classes of continuous 1-cocycles on $\mathcal{U}$ is denoted

$$\check{\mathrm{H}}^1_{\mathrm{cont}}(\mathcal{U}; G).$$

If $\mathcal{V}$ is an open covering of X such that $\mathcal{V} \prec \mathcal{U}$, there is a canonical restriction map

$$\check{\mathrm{H}}^1_{\mathrm{cont}}(\mathcal{U}; G) \to \check{\mathrm{H}}^1_{\mathrm{cont}}(\mathcal{V}; G).$$

These restriction maps turn the family $\check{\mathrm{H}}^1_{\mathrm{cont}}(\mathcal{U};G)$ ($\mathcal{U}$ open covering of X) into an inductive system and we set

$$\check{\mathrm{H}}^1_{\mathrm{cont}}(X;G) = \varinjlim_{\mathcal{U}} \check{\mathrm{H}}^1_{\mathrm{cont}}(\mathcal{U};G).$$

Proposition 3.5.2. *For any topological space B, there is a canonical bijection*

$$\mathrm{Isom}(\mathcal{V}\mathrm{ect}^r_{\mathbb{R}}(B)) \xrightarrow{\sim} \check{\mathrm{H}}^1_{cont}(B;\mathrm{GL}_r(\mathbb{R})).$$

Proof. Let E be a real vector bundle with base B and rank r. By definition, there is an open covering $\mathcal{U}$ of B such that $E_{|U}$ has a continuous frame for any $U \in \mathcal{U}$. This gives us a family of trivializations

$$\varphi_U : E_{|U} \xrightarrow{\sim} \mathbb{R}^r \times U \qquad (U \in \mathcal{U}).$$

For any U, $V \in \mathcal{U}$,

$$\varphi_V \circ \varphi_U^{-1} : \mathbb{R}^r \times (U \cap V) \to \mathbb{R}^r \times (U \cap V)$$

is an isomorphism of real vector bundles. Therefore, the map

$$v \mapsto q_2\left[(\varphi_V \circ \varphi_U^{-1})(v,x)\right]$$

defines an element $\psi_{VU}(x)$ of $\mathrm{GL}_r(\mathbb{R})$. Moreover, it is clear that

$$\psi_{VU} : U \cap V \to \mathrm{GL}_r(\mathbb{R})$$

is continuous and that

$$\psi_{WV}(x)\psi_{VU}(x) = \psi_{WU}(x)$$

for any $x \in U \cap V \cap W$. It follows that $(\psi_{VU})_{U,V\in\mathcal{U}}$ is a continuous 1-cocycle on $\mathcal{U}$ with values in $\mathrm{GL}_r(\mathbb{R})$. We leave it to the reader to check that its class in $\check{\mathrm{H}}^1_{\mathrm{cont}}(X;\mathrm{GL}_r(\mathbb{R}))$ depends only on the isomorphy class of E. As a consequence, we get a well-defined map

$$\mathrm{Isom}(\mathcal{V}\mathrm{ect}^r_{\mathbb{R}}(B)) \to \check{\mathrm{H}}^1_{\mathrm{cont}}(X;\mathrm{GL}_r(\mathbb{R})).$$

Its injectivity is almost obvious. To prove its surjectivity, it is sufficient to consider a continuous 1-cocycle $(\psi_{VU})_{U,V\in\mathcal{U}}$ on $\mathcal{U}$ with values in $\mathrm{GL}_r(\mathbb{R})$ and to show that its equivalence class is the class associated to the real vector bundle obtained by gluing together the family of trivial bundles

$$\mathbb{R}^r \times U$$

through the transition isomorphisms

$$\begin{aligned} \mathbb{R}^r \times (U \cap V) &\to \mathbb{R}^r \times (U \cap V) \\ (v, x) &\mapsto (\psi_{VU}(x)v, x). \end{aligned}$$

Details are left to the reader. □

Lemma 3.5.3. *For any $x \in \mathbb{R}^*$, set*

$$s(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x < 0. \end{cases}$$

Then,

$$s : \mathbb{R}^* \to \mathbb{Z}_2$$

is a morphism of groups and the sequence of abelian groups

$$0 \to \mathbb{R} \xrightarrow{\exp} \mathbb{R}^* \xrightarrow{s} \mathbb{Z}_2 \to 0$$

is exact. Moreover, for any topological space X, this sequence induces the exact sequence of sheaves

$$0 \to \mathcal{C}_X^{\mathbb{R}} \to \mathcal{C}_X^{\mathbb{R}^*} \to (\mathbb{Z}_2)_X \to 0$$

where $\mathcal{C}_X^{\mathbb{R}}$ denotes the sheaf of real valued continuous functions.

Proof. Direct. □

Proposition 3.5.4. *Let B be a topological space. The exact sequence*

$$0 \to \mathcal{C}_B^{\mathbb{R}} \to \mathcal{C}_B^{\mathbb{R}^*} \to (\mathbb{Z}_2)_B \to 0$$

induces an isomorphism

$$\mathrm{H}^1(B; \mathcal{C}_B^{\mathbb{R}^*}) \simeq \mathrm{H}^1(B; \mathbb{Z}_2).$$

There is a bijection

$$\check{\mathrm{H}}^1_{cont}(B; \mathrm{GL}_1(\mathbb{R})) \simeq \mathrm{H}^1(B; \mathcal{C}_B^{\mathbb{R}^*}).$$

The associated bijection

$$\mathrm{Isom}(\mathcal{V}\mathrm{ect}^1_{\mathbb{R}}(B)) \simeq \mathrm{H}^1(B; \mathbb{Z}_2)$$

may be realized by the map

$$L \mapsto e_L^{\mathbb{Z}_2}.$$

Proof. Taking the long exact sequence of cohomology associated to the sequence of sheaves

$$0 \to \mathcal{C}_B^{\mathbb{R}} \to \mathcal{C}_B^{\mathbb{R}^*} \to (\mathbb{Z}_2)_B \to 0,$$

we get the exact sequence

$$\mathrm{H}^1(B;\mathcal{C}_B^{\mathbb{R}}) \to \mathrm{H}^1(B;\mathcal{C}_B^{\mathbb{R}^*}) \to \mathrm{H}^1(B;\mathbb{Z}_2) \to \mathrm{H}^2(B;\mathcal{C}_B^{\mathbb{R}}).$$

Since $\mathcal{C}_B^{\mathbb{R}}$ is soft, the first and last terms vanish. Hence the isomorphism

$$\mathrm{H}^1(B;\mathcal{C}_B^{\mathbb{R}^*}) \simeq \mathrm{H}^1(B;\mathbb{Z}_2).$$

Clearly, there is a bijection

$$\check{\mathrm{H}}^1_{\mathrm{cont}}(B;\mathrm{GL}_1(\mathbb{R})) \simeq \check{\mathrm{H}}^1(B;\mathcal{C}_B^{\mathbb{R}^*})$$

and using the isomorphism between Čech cohomology and ordinary cohomology, we get a bijection

$$\check{\mathrm{H}}^1_{\mathrm{cont}}(B;\mathrm{GL}_1(\mathbb{R})) \simeq \mathrm{H}^1(B;\mathcal{C}_B^{\mathbb{R}^*}).$$

Combining this with the bijection of Proposition 3.5.2, we get a bijection

$$\mathrm{Isom}(\mathcal{V}\mathrm{ect}^1_{\mathbb{R}}(B)) \simeq \mathrm{H}^1(B;\mathbb{Z}_2). \tag{*}$$

What remains to prove is that this bijection may be realized by the Euler class. Since this bijection is clearly compatible with the pull-back of bundles and cohomology classes, Theorem 3.3.13 shows that the result will be true if the image of U_∞ in $\mathrm{H}^1(\mathbb{P}_\infty(\mathbb{R}),\mathbb{Z}_2)$ by (*) is the Euler class of U_∞. We know that the restriction map

$$\mathrm{H}^1(\mathbb{P}_m(\mathbb{R});\mathbb{Z}_2) \to \mathrm{H}^1(\mathbb{P}_n(\mathbb{R});\mathbb{Z}_2)$$

is an isomorphism for $m \geq n \geq 1$. It follows that the restriction map

$$\mathrm{H}^1(\mathbb{P}_\infty(\mathbb{R});\mathbb{Z}_2) \to \mathrm{H}^1(\mathbb{P}_1(\mathbb{R});\mathbb{Z}_2)$$

is also an isomorphism. Hence, we are reduced to prove that the image of $\mathbb{U}_1(\mathbb{R})$ by (*) in $\mathrm{H}^1(\mathbb{P}_1(\mathbb{R});\mathbb{Z}_2)$ is $e^{\mathbb{Z}_2}_{\mathbb{U}_1(\mathbb{R})}$. Set $V_0 = \{[x_0,x_1] : x_0 \neq 0\}$ and $V_1 = \{[x_0,x_1] : x_1 \neq 0\}$. Clearly, V_0, V_1 are open subsets of $\mathbb{P}_1(\mathbb{R})$ which are homeomorphic to $\mathbb{R}$. Moreover, $\mathbb{P}_1(\mathbb{R}) = V_0 \cup V_1$ and $V_0 \cap V_1$ is homeomorphic to $\mathbb{R}^*$. The line bundle $\mathbb{U}_1(\mathbb{R})_{|V_0}$ has a continuous frame given by

$$[x_0,x_1] \mapsto \left((1,\frac{x_1}{x_0}),[x_0,x_1]\right).$$

Similarly, the line bundle $\mathbb{U}_1(\mathbb{R})_{|V_1}$ has a continuous frame given by

$$[x_0, x_1] \mapsto \left((\frac{x_0}{x_1}, 1), [x_0, x_1] \right).$$

The continuous 1-cocycle on $\mathcal{V} = \{V_0, V_1\}$ with values in $\mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^*$ associated to U_1 is thus given by

$$\psi_{V_0V_0} = 1, \qquad \psi_{V_1V_1} = 1, \qquad \psi_{V_1V_0} = \frac{x_1}{x_0}, \qquad \psi_{V_0V_1} = \frac{x_0}{x_1}.$$

Its image c in $\check{\mathrm{H}}^1(\mathcal{V}; \mathbb{Z}_2)$ is thus the class of the 1-cocycle

$$\psi'_{V_0V_0} = 0, \qquad \psi'_{V_1V_1} = 0, \qquad \psi'_{V_1V_0} = s(\frac{x_1}{x_0}), \qquad \psi'_{V_0V_1} = s(\frac{x_0}{x_1}).$$

Let $(\psi'_{V_0}, \psi'_{V_1})$ be an element of $\check{C}^0(\mathcal{V}; \mathbb{Z}_2)$. Then, ψ'_{V_0} (resp. ψ'_{V_1}) is constant on V_0 (resp. V_1) and

$$d(\psi'_{V_0}, \psi'_{V_1})_{V_1V_0} = \psi'_{V_0|V_1 \cap V_0} - \psi'_{V_1|V_1 \cap V_0}$$

is also constant on $V_1 \cap V_0$. Since $\psi'_{V_1V_0}$ is not constant on $V_1 \cap V_0$, it follows that $c \neq 0$. Using the fact that $\check{\mathrm{H}}^1(\mathcal{V}; \mathbb{Z}_2) \simeq \mathrm{H}^1(\mathbb{P}_1(\mathbb{R}); \mathbb{Z}_2) \simeq \mathbb{Z}_2$, it follows that $c = e^{\mathbb{Z}_2}_{\mathbb{U}_1(\mathbb{R})}$. ☐

Corollary 3.5.5. *Assume B is a topological space. Then, a real line bundle L on a B is trivial if and only if*

$$e^{\mathbb{Z}_2}_L = 0.$$

In particular, a real vector bundle E on B is orientable if and only if

$$w_1(E) = 0$$

Proof. Since a real vector bundle E is orientable if and only if the line bundle $\mathrm{Det}(E)$ is trivializable, the result follows directly from the preceding proposition combined with Corollary 3.2.9. ☐

4

Characteristic classes of complex vector bundles

In this chapter as in the previous one, all topological spaces are implicitly assumed to be paracompact.

4.1 Generalities on complex vector bundles

Definition 4.1.1. A *complex vector bundle of rank* r is the data of a continuous map

$$p_E : E \to B_E$$

between topological spaces together with structures of complex vector spaces on each fiber $E_b = p_E^{-1}(b)$ $(b \in B_E)$ of P_E. These data being such that for any $b \in B$ there is a neighborhood U of b in B and a family $(e_1, \cdots, e_r)$ of continuous sections of $p_{E|p_E^{-1}(U)} : p_E^{-1}(U) \to U$ with the property that $(e_1(b'), \cdots, e_r(b'))$ is a basis of $p_E^{-1}(b')$ for any $b' \in U$.

As seen from the preceding definition, the notion of complex vector bundle is completely similar to that of real vector bundle. We just have to replace $\mathbb{R}$-linearity with $\mathbb{C}$-linearity when appropriate. This is why we will not define in details the vocabulary concerning complex vector bundles, assuming the reader can adapt easily what has been done for real vector bundles.

Lemma 4.1.2. *Let $u : E \to E$ be a morphism of complex vector bundles. Denote $E_\mathbb{R}$ the real vector bundle associated to E and*

$$u_\mathbb{R} : E_\mathbb{R} \to E_\mathbb{R}$$

the morphism of real vector bundles associated to u. Then,

$$\det_\mathbb{R} u_\mathbb{R} = |\det_\mathbb{C} u|^2.$$

Proof. Let e be an eigenvector of u and denote λ the associated eigenvalue. Choose F such that $E = \rangle e \langle \oplus F$ and denote $p : E \to F$ the associated projection. Denote $\mu : \rangle e \langle \to \rangle e \langle$ the multiplication by λ and $v : F \to F$ the map $p \circ u_{|F}$. Then,

$$\det_\mathbb{C} u = \lambda \det_\mathbb{C} v$$

and

$$\det_\mathbb{R} u_\mathbb{R} = \det_\mathbb{R} \mu_\mathbb{R} \cdot \det_\mathbb{R} v_\mathbb{R}.$$

Therefore, an induction argument reduces the problem to the case where $\dim_\mathbb{C} E = 1$. In this case, if e is a non-zero vector of E and $u(e) = \lambda e$, we have

$$\det_\mathbb{C} u = \lambda.$$

Since (e, ie) is a basis of $E_\mathbb{R}$ and the matrix of $u_\mathbb{R}$ in this basis is

$$\begin{pmatrix} \Re\lambda & -\Im\lambda \\ \Im\lambda & \Re\lambda \end{pmatrix}$$

we have $\det_\mathbb{R} u_\mathbb{R} = \Re\lambda^2 + \Im\lambda^2 = |\lambda|^2$. The conclusion follows. □

Corollary 4.1.3. *Let E be a complex vector bundle of rank r on B. Then, the underlying real vector bundle $E_\mathbb{R}$ is canonically oriented. If $(e_1, \cdots, e_r)$ is a continuous local frame of E, then $(e_1, ie_1, \cdots, e_r, ie_r)$ is a continuous oriented local frame of $E_\mathbb{R}$. In particular, the Euler class $e^\mathbb{Z}_{E_\mathbb{R}} \in \mathrm{H}^{2r}(B; \mathbb{Z})$ is well-defined.*

Exercise 4.1.4. *Denote $\mathbb{P}_n(\mathbb{C})$ the complex projective space and $\mathbb{U}_n(\mathbb{C})$ the complex universal bundle. Set $\xi = e^\mathbb{Z}_{\mathbb{U}_n(\mathbb{C})} \in \mathrm{H}^2(\mathbb{P}_n(\mathbb{C}); \mathbb{Z})$. Then,*

$$\mathrm{H}^{\cdot}(\mathbb{P}_n(\mathbb{C}); \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}\xi \oplus \cdots \oplus \mathbb{Z}\xi^n.$$

In particular,

$$\chi(\mathbb{P}_n(\mathbb{C})) = n + 1.$$

Solution. Work as in Exercise 2.5.6. □

Proposition 4.1.5. *Let E be a complex vector bundle of rank r on B. Denote $\pi : P(E) \to B$ the associated complex projective bundle. Let $U(E)$ be the universal complex line bundle on $P(E)$ and let $\xi = e^{\mathbb{Z}}_{U(E)} \in \mathrm{H}^2(P(E);\mathbb{Z})$ be its Euler class. Then*

$$\mathrm{H}^{\cdot}(P(E);\mathbb{Z})$$

is a free $\mathrm{H}^{\cdot}(B;\mathbb{Z})$-module of rank r with

$$1, \xi, \xi^2, \cdots, \xi^{r-1}$$

as basis.

Proof. Work as in the proof of Proposition 3.1.2. □

Corollary 4.1.6. *Let E be a complex vector bundle of rank r on the paracompact base X. Then, there is a proper map*

$$f : Y \to X$$

for which the canonical map

$$f^* : \mathrm{H}^{\cdot}(X;\mathbb{Z}) \to \mathrm{H}^{\cdot}(Y;\mathbb{Z})$$

is injective and such that

$$f^{-1}(E) \simeq L_1 \oplus \cdots \oplus L_r$$

with $L_1, \cdots, L_r$ complex line bundles on Y.

Proof. Work as in the proof of Proposition 3.2.1. □

4.2 Chern classes

Definition 4.2.1. Using the notations of Proposition 4.1.5, we define the *Chern classes* of E as the classes $c_1(E) \in \mathrm{H}^2(B;\mathbb{Z}), \cdots, c_r(E) \in \mathrm{H}^{2r}(B;\mathbb{Z})$ characterized by the relation

$$\xi^r = \pi^*(c_1(E)) \smile \xi^{r-1} - \pi^*(c_2(E)) \smile \xi^{r-2} + \cdots + (-1)^r \pi^* \smile (c_r(E)).$$

By convention, we extend the preceding definition by setting

$$c_0(E) = 1, \quad c_k(E) = 0 \quad (k > r).$$

As for Stiefel-Whitney classes, we also define the *total Chern class* $c(E)$ as the sum

$$\sum_k c_k(E) \in \mathrm{H}^{\cdot}(B;\mathbb{Z}).$$

Remark 4.2.2. Note that $c(E)$ is in fact an element of

$$\mathrm{H}^{\mathrm{ev}}(B;\mathbb{Z}) := \bigoplus_{k\in\mathbb{N}} \mathrm{H}^{2k}(B;\mathbb{Z})$$

which is a commutative subalgebra of $\mathrm{H}^{\cdot}(B;\mathbb{Z})$.

Proposition 4.2.3. *Assume E, F are complex vector bundles on B of rank r, s and let $f : B' \to B$ be a continuous map. Then,*

(a) $c(f^{-1}(E)) = f^*(c(E))$;

(b) $c(E \oplus F) = c(E) \smile c(F)$;

(c) $c_r(E) = e^{\mathbb{Z}}_{E_{\mathbb{R}}}$.

Proof. Work as in Propositions 3.1.4, 3.1.5 and 3.2.3. ☐

Definition 4.2.4. Let E be complex vector bundle with base B. We denote $\overline{E}$ the complex vector bundle obtained from E by changing the $\mathbb{C}$-vector space structure of each fiber into its conjugate one.

Proposition 4.2.5. *Let E be a complex vector bundle with rank r and base B. Then,*

$$c(E^*) = c(\overline{E}) = (-1)^r c(E).$$

Proof. Endowing E with a Hermitian structure, one sees easily that

$$E^* \simeq \overline{E}$$

and the first equality follows. To get the second one, we may use the splitting principle and treat only the case where E is a complex line bundle. In this case, we have only to show that

$$e^{Z}_{\overline{E}} = -e^{Z}_{E}.$$

In other words, we have to show that that the canonical orientations o and $\overline{o}$ of $E_{\mathbb{R}}$ induced by the complex structure of E and $\overline{E}$ are opposite. This follows from the fact that if e is a complex local frame of E, then e, ie is a positively oriented local frame for o and $e, -ie$ is a positively oriented local frame for $\overline{o}$. ☐

Exercise 4.2.6. *With the notations of Exercise 4.1.4, show that*

$$c(T\mathbb{P}_n(\mathbb{C})) = (1+\xi)^{n+1}$$

and deduce from this formula that

$$\int_{\mathbb{P}_n(\mathbb{C})} \xi^n = 1.$$

Solution. The first relation is obtained by working as in Exercise 3.2.17. From this relation we deduce that

$$\epsilon_{\mathbb{P}_n(\mathbb{C})} = c_n(T\mathbb{P}_n(\mathbb{C})) = (n+1)\xi^n.$$

Therefore, using the index theorem for compact manifolds, we get

$$\int_{\mathbb{P}_n(\mathbb{C})} (n+1)\xi^n = \int_{\mathbb{P}_n(\mathbb{C})} \epsilon_{\mathbb{P}_n(\mathbb{C})} = \chi(\mathbb{P}_n(\mathbb{C})) = n+1.$$

The conclusion follows. □

Proposition 4.2.7. *Denote $G_{\infty,r}(\mathbb{C})$ the complex infinite Grassmannian of rank r and $U_{\infty,r}(\mathbb{C})$ the associated universal complex vector bundle of rank r. Assume E is a complex vector bundle of rank r on B. Then, there is a continuous map $f : B \to G_{\infty,r}(\mathbb{C})$ such that*

$$E \simeq f^{-1}U_{\infty,r}(\mathbb{C}).$$

Moreover, such a map is unique up to homotopy.

Proof. Work as for Corollary 3.3.14. □

Corollary 4.2.8. *Characteristic classes with values in the abelian group M of complex vector bundles of rank r are in bijection with*

$$\mathrm{H}^{\cdot}(G_{\infty,r}(\mathbb{C}); M).$$

Proposition 4.2.9. *The morphism of rings*

$$\mathbb{Z}[C_1, \cdots, C_r] \to \mathrm{H}^{\cdot}(G_{\infty,r}(\mathbb{C}); \mathbb{Z})$$

defined by sending C_k to $c_k(U_{\infty,r}(\mathbb{C}))$ is an isomorphism.

Proof. Work as for Proposition 3.4.7. □

Proposition 4.2.10. *The exact sequence of sheaves*

$$0 \to \mathbb{Z}_B \xrightarrow{2i\pi} \mathcal{C}_B \to \mathcal{C}_B^* \to 0$$

induces an isomorphism

$$\mathrm{H}^1(B; \mathcal{C}_B^*) \simeq \mathrm{H}^2(B; \mathbb{Z}_B).$$

By this isomorphism, the class of a complex line bundle L is sent to $c_1(L)$.

Proof. Working as in the proof of Proposition 3.5.4, we may assume $B = \mathbb{P}_1(\mathbb{C})$, $L = U_1(\mathbb{C})$. Set $V_0 = \{[z_0, z_1] : z_0 \neq 0\}$, $V_1 = \{[z_0, z_1] : z_1 \neq 0\}$. Define $v_0 : V_0 \to \mathbb{C}$ and $v_1 : V_1 \to \mathbb{C}$ by setting

$$v_0 = \frac{z_1}{z_0}, \qquad v_1 = \frac{z_0}{z_1}.$$

Clearly, $U_1(\mathbb{C})_{|V_0}$ has a continuous frame given by s_0 defined by

$$[z_0, z_1] \mapsto \left((1, \frac{z_1}{z_0}), [z_0, z_1]\right).$$

Similarly, $U_1(\mathbb{C})_{|V_1}$ has a continuous frame given by s_1 defined by

$$[z_0, z_1] \mapsto \left((\frac{z_0}{z_1}, 1), [z_0, z_1]\right).$$

Since

$$s_{0|V_0 \cap V_1} = v_0 s_{1|V_0 \cap V_1},$$

the class of L in $\mathrm{H}^1(\mathbb{P}_1(\mathbb{C}), \mathcal{C}^*_{\mathbb{P}_1(\mathbb{C})})$ is the image of v_0 by the coboundary operator

$$\mathrm{H}^0(V_0 \cap V_1; \mathcal{C}^*_{\mathbb{P}_1(\mathbb{C})}) \to \mathrm{H}^1(\mathbb{P}_1(\mathbb{C}); \mathcal{C}^*_{\mathbb{P}_1(\mathbb{C})})$$

of the Mayer-Vietoris sequence associated to the decomposition $\mathbb{P}_1(\mathbb{C}) = V_0 \cup V_1$. Define $\mathcal{Z}^1_\infty$ as the kernel of the de Rham differential

$$d^1 : \mathcal{C}^1_\infty \to \mathcal{C}^2_\infty$$

and

$$\mathrm{dlog} : \mathcal{C}^*_\infty \to \mathcal{Z}^1_\infty$$

by setting

$$\mathrm{dlog}(f) = \frac{df}{f}.$$

The morphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "2i\pi"] \arrow[d, "2i\pi"] & \mathcal{C}_\infty \arrow[r, "\exp"] \arrow[d, "\mathrm{id}"] & \mathcal{C}^*_\infty \arrow[r] \arrow[d, "\mathrm{dlog}"] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r] & \mathcal{C}_\infty \arrow[r, "d"'] & \mathcal{Z}^1_\infty \arrow[r] & 0
\end{tikzcd}$$

induces the commutative diagram of coboundary operators

$$\begin{tikzcd}
\mathrm{H}^1(\mathbb{P}_1(\mathbb{C}); \mathcal{C}^*_\infty) \arrow[r, squiggly] \arrow[d, "\mathrm{dlog}"'] & \mathrm{H}^2(\mathbb{P}_1(\mathbb{C}); \mathbb{Z}) \arrow[d, "2i\pi"] \\
\mathrm{H}^1(\mathbb{P}_1(\mathbb{C}); \mathcal{Z}^1_\infty) \arrow[r, squiggly] & \mathrm{H}^2(\mathbb{P}_1(\mathbb{C}); \mathbb{C})
\end{tikzcd}$$

We have to show that the image of $[L]$ in $\mathrm{H}^2(\mathbb{P}_1(\mathbb{C});\mathbb{Z})$ is a generator. Hence it is sufficient to show that

$$\int_{\mathbb{P}_1(\mathbb{C})} \mathrm{dlog}([L]) = 2i\pi.$$

To compute $\mathrm{dlog}([L])$, let us proceed as follows. First, the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^0(V_0\cap V_1;\mathcal{C}^*_\infty) & \longrightarrow & \mathrm{H}^1(\mathbb{P}_1(\mathbb{C});\mathcal{C}^*_\infty) \\ {\scriptstyle\mathrm{dlog}}\downarrow & & \downarrow{\scriptstyle\mathrm{dlog}} \\ \mathrm{H}^0(V_0\cap V_1;\mathcal{Z}^1_\infty) & \longrightarrow & \mathrm{H}^1(\mathbb{P}_1(\mathbb{C});\mathcal{Z}^1_\infty) \end{array}$$

for the coboundary operators of the Mayer-Vietoris sequences of $\mathcal{C}^*_\infty$ and $\mathcal{Z}^1_\infty$, shows that $\mathrm{dlog}[L]$ is the image of

$$\frac{dv_0}{v_0} \in \mathrm{H}^0(V_0\cap V_1;\mathcal{Z}^1_\infty)$$

in $\mathrm{H}^1(\mathbb{P}_1(\mathbb{C});\mathcal{Z}^1_\infty)$. To compute this image, we will use the soft resolution

$$0 \to \mathcal{Z}^1_\infty \to \mathcal{C}^1_\infty \xrightarrow{d} \mathcal{C}^2_\infty \to 0$$

of $\mathcal{Z}^1_\infty$. We have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to & \Gamma(\mathbb{P}_1(\mathbb{C});\mathcal{C}^1_\infty) & \to & \Gamma(V_0;\mathcal{C}^1_\infty)\oplus\Gamma(V_1;\mathcal{C}^1_\infty) & \to & \Gamma(V_0\cap V_1;\mathcal{C}^1_\infty) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \Gamma(\mathbb{P}_1(\mathbb{C});\mathcal{C}^2_\infty) & \to & \Gamma(V_0;\mathcal{C}^2_\infty)\oplus\Gamma(V_1;\mathcal{C}^2_\infty) & \to & \Gamma(V_0\cap V_1;\mathcal{C}^2_\infty) & \to 0 \end{array}$$

To obtain the image β of $\dfrac{dv_0}{v_0}$ in $\Gamma(\mathbb{P}_1(\mathbb{C});\mathcal{C}^2_\infty)/d\Gamma(\mathbb{P}_1(\mathbb{C});\mathcal{C}^1_\infty)$, we have first to find $(\alpha,\alpha')\in\Gamma(V_0;\mathcal{C}^1_\infty)\oplus\Gamma(V_1;\mathcal{C}^1_\infty)$ such that $\alpha_{|V_0\cap V_1} - \alpha'_{|V_0\cap V_1} = \dfrac{dv_0}{v_0}$ and then to use the relations $\beta_{|V_0} = d\alpha$, $\beta_{|V_1} = d\alpha'$. Let φ be a C_∞-function on $\mathbb{C}$ which is zero for $|z|\leq 1/4$ and 1 for $|z|\geq 3/4$. Clearly, we may take

$$\alpha = \varphi(v_0)\frac{dv_0}{v_0}, \qquad \alpha' = -(1-\varphi(v_0))\frac{dv_0}{v_0}.$$

Then,

$$d\alpha_{|V_0\cap V_1} = d\alpha'_{|V_0\cap V_1}$$

and these forms have compact supports. Moreover, β is equal to $d\alpha$ on $V_0\cap V_1$ and 0 on $\mathbb{P}_1(\mathbb{C})\setminus(V_0\cap V_1)$. Therefore, denoting $D(0,1)$ the unit disk in $\mathbb{C}$, we have

$$\begin{aligned}\int_{\mathbb{P}_1(\mathbb{C})} \beta &= \int_{V_0} d\alpha = \int_{\mathbb{C}} d(\varphi(z)\frac{dz}{z}) \\ &= \int_{D(0,1)} d(\varphi(z)\frac{dz}{z}) = \int_{\partial D(0,1)} \varphi(z)\frac{dz}{z} \\ &= \int_{\partial D(0,1)} \frac{dz}{z} = 2i\pi\end{aligned}$$

where the last formula follows from Cauchy's theorem. □

Proposition 4.2.11. *Assume E (resp. F) is a complex vector bundle of rank r (resp. s) on B. Then,*

$$c(E \otimes F) = T_{r,s}(c_1(E), \dots, c_r(E), c_1(F), \dots, c_s(F)).$$

Proof. Working as in the proof of Proposition 3.2.12, we see that it is sufficient to treat the case $r = s = 1$. In this case, we have only to prove that

$$c_1(E \otimes F) = c_1(E) + c_1(F).$$

This follows easily from the preceding proposition. □

4.3 Chern-Weil construction

In this section, $p : E \to B$ will denote a differentiable complex vector bundle of rank r on the paracompact differential manifold B. As usual, for such a bundle, $C^p_\infty(B; E)$ will denote the space of differentiable p-forms with values in E (i.e. the differentiable sections of $\bigwedge^p T^*X^{\mathbb{C}} \otimes_{\mathbb{C}} E$).

A $\mathbb{C}$-*linear connection* on E is the data of a $\mathbb{C}$-linear operator

$$\nabla : C^0_\infty(B; E) \to C^1_\infty(B; E)$$

satisfying Leibnitz rule

$$\nabla(fs) = (df)s + f(\nabla s)$$

for any $f \in C_\infty(B)$, $s \in C_\infty(B; E)$. Such an operator gives rise to a family of operators

$$\nabla^p : C^p_\infty(B; E) \to C^{p+1}_\infty(B; E)$$

which is uniquely characterized by the fact that

$$\nabla^p(\alpha s) = (d\alpha)s + (-1)^p \alpha \wedge (\nabla s)$$

for any $\alpha \in C^p_\infty(B;E)$ and any $s \in C^0_\infty(B;E)$. Let $e_1, \cdots, e_r$ be a differentiable frame of E on the open subset U of B. Then,

$$\bigtriangledown e_j = \sum_{k=1}^{r} \omega_{kj} e_k$$

where ω is a $r \times r$ matrix of differential 1-forms on U. We call ω the *matrix of the connection* $\bigtriangledown$ *in the local frame* $e_1, \cdots, e_r$. If s is a differentiable section of E on U, then s may be written in a unique way as

$$s = \sum_{j=1}^{r} s_j e_j.$$

For such an s, we have

$$\bigtriangledown s = \sum_{j=1}^{r} (ds_j) e_j + \sum_{j=1}^{r} \sum_{k=1}^{r} s_j \omega_{kj} e_k.$$

Hence,

$$(\bigtriangledown s)_j = ds_j + \sum_{k=1}^{r} \omega_{jk} s_k.$$

Moreover,

$$\begin{aligned}
&\bigtriangledown^1 \cdot \bigtriangledown^0 s \\
&\quad = \bigtriangledown^1 (\sum_{j=1}^{r} (ds_j + \sum_{k=1}^{r} \omega_{jk} s_k) e_j) \\
&\quad = \sum_{j=1}^{r} \sum_{k=1}^{r} ((d\omega_{jk}) s_k - \omega_{jk} \wedge ds_k) e_j - (ds_j + \sum_{k=1}^{r} \omega_{jk} s_k) \wedge \sum_{l=1}^{r} \omega_{lj} e_l.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\bigtriangledown^1 \cdot \bigtriangledown^0 s)_j &= \sum_{k=1}^{r} (d\omega_{jk}) s_k - \omega_{jk} \wedge ds_k - \sum_{l=1}^{r} (ds_l + \sum_{k=1}^{r} \omega_{lk} s_k) \wedge \omega_{jl} \\
&= \sum_{k=1}^{r} (d\omega_{jk}) s_k - \sum_{l=1}^{r} \sum_{k=1}^{r} \omega_{lk} \wedge \omega_{jl} s_k \\
&= \sum_{k=1}^{r} \left(d\omega_{jk} + \sum_{l=1}^{r} \omega_{jl} \wedge \omega_{lk} \right) s_k.
\end{aligned}$$

In particular, $\bigtriangledown^1 \cdot \bigtriangledown^0 : C^0_\infty(B;E) \to C^2_\infty(B;E)$ is $C_\infty(E)$-linear and hence comes from a morphism of vector bundles

$$K : E \to \bigwedge^2 T^* X^{\mathbb{C}} \otimes_{\mathbb{C}} E$$

whose matrix in the local frame $e_1, \cdots, e_r$ is

$$\Omega = d\omega + \omega \wedge \omega. \tag{*}$$

The morphism K is called *the curvature of the connection* $\bigtriangledown$. Note that more generally,

$$\begin{aligned}\bigtriangledown^{p+1} \cdot \bigtriangledown^{p}(\alpha \circ s) &= \bigtriangledown^{p+1}((d\alpha)s + (-1)^p \alpha \wedge (\bigtriangledown s)) \\ &= (-1)^{p+1}(d\alpha) \wedge (\bigtriangledown s) + (-1)^p (d\alpha) \wedge (\bigtriangledown s) \\ &\quad + (-1)^{2p} \alpha \wedge (\bigtriangledown^1 \cdot \bigtriangledown^0 (s)) \\ &= \alpha \wedge K(s).\end{aligned}$$

Let $f : B' \to B$ be a differentiable map and let $\bigtriangledown$ be a $\mathbb{C}$-linear connection on E. Clearly, $f^{-1}(E)$ is a differentiable complex bundle on B'. We denote $f^{-1}\bigtriangledown$ the $\mathbb{C}$-linear connection on $f^{-1}E$ characterized by the fact that its matrix with respect to the differentiable frame $e_1 \circ f, \cdots, e_r \circ f$ of $f^{-1}E$ on $f^{-1}(U)$ associated to a differentiable frame $e_1, \cdots, e_r$ of E on U is the pull-back by f of the matrix of $\bigtriangledown$ in $e_1, \cdots, e_r$ (i.e. we set

$$\omega_{f^{-1}\bigtriangledown} = f^* \omega_{\bigtriangledown}).$$

We leave it to the reader to check that this definition is meaningful. From this definition and formula (*), it follows easily that, if $\Omega_{f^{-1}\bigtriangledown}$ (resp. $\Omega_{\bigtriangledown}$) denotes the matrix of the curvature of $f^{-1}\bigtriangledown$ (resp. $\bigtriangledown$) in the local frame $e_1 \circ f, \cdots, e_r \circ f$ (resp. $e_1, \cdots, e_r$) of $f^{-1}(E)$ (resp. E), then

$$\Omega_{f^{-1}\bigtriangledown} = f^* \Omega_{\bigtriangledown}.$$

Denote $M_r(\mathbb{C})$ the algebra of $r \times r$ matrices of complex numbers. An *invariant homogeneous polynomial* of degree k on $M_r(\mathbb{C})$ is a homogeneous polynomial map

$$P : M_r(\mathbb{C}) \to \mathbb{C}$$

of degree k such that

$$P(T^{-1}AT) = P(A)$$

for any $T \in \mathrm{GL}_r(\mathbb{C})$ and any $A \in M_r(\mathbb{C})$. (Examples of such invariant polynomials are given by the determinant (degree r) or the trace (degree 1)). Let P be an invariant homogeneous polynomial of degree k on $M_r(\mathbb{C})$ and let $\bigtriangledown$ be a $\mathbb{C}$-linear connection on E. Denote $\Omega_{\bigtriangledown,e}$ the matrix of the curvature of $\bigtriangledown$ in a frame $e = (e_1, \cdots, e_r)$ of E on the open subset U of B. Since $C^{\mathrm{ev}}_{\infty}(U; E)$ is a commutative $\mathbb{C}$-algebra, the expression

$$P(\Omega_{\bigtriangledown,e})$$

is a well-defined differential form of degree $2k$ on U. Moreover, since P is invariant, $P(\Omega_{\bigtriangledown,e})$ does not depend on the frame e. It follows that there is a unique differential form $P(K_{\bigtriangledown})$ characterized by the fact that

$$P(K_{\bigtriangledown})_{|U} = P(\Omega_{\bigtriangledown,e})$$

for any local frame $e = (e_1, \cdots, e_r)$ of E. Moreover, by construction, we have

$$P(K_{f^{-1}\bigtriangledown}) = f^* P(K_{\bigtriangledown}).$$

Proposition 4.3.1. *For any $\mathbb{C}$-linear connection $\bigtriangledown$ on E and any invariant homogeneous polynomial P on $M_r(\mathbb{C})$,*

$$P(K_{\bigtriangledown})$$

is a closed differential form of degree $2k$.

Proof. Let us define the matrix $P'(A)$ by setting

$$P'(A)_{jk} = \frac{\partial P(A)}{\partial A_{kj}}.$$

Then, for any $H \in M_r(\mathbb{C})$, we have

$$\begin{aligned} L_H P(A) &= \sum_{j=1}^{r} \sum_{k=1}^{r} \frac{\partial P(A)}{\partial A_{kj}} H_{kj} \\ &= \operatorname{tr}(P'(A)H) \end{aligned}$$

where L_H denotes the derivative in the direction H. The fact that P is invariant entails that $P'(A)$ commutes with A. As a matter of fact, let $H \in M_r(\mathbb{C})$. From the relation

$$P(A(I + tH)) = P((I + tH)A)$$

which holds for any $t \in \mathbb{R}$, we deduce by derivation that

$$\operatorname{tr}(P'(A)AH) = \operatorname{tr}(P'(A)HA) = tr(AP'(A)H).$$

Since H is arbitrary, we see that

$$P'(A)A = AP'(A).$$

Let ω (resp. Ω) be the matrix (resp. the curvature matrix) of the connection $\bigtriangledown$ with respect to a differentiable frame $e_1, \cdots, e_r$ of E on U. A simple computation shows that

$$dP(\Omega) = \sum_{j=1}^{r} \sum_{k=1}^{r} P'(\Omega)_{jk} \wedge d\Omega_{kj} = \operatorname{tr}(P'(\Omega) \wedge d\Omega).$$

Since

$$\Omega = d\omega + \omega \wedge \omega,$$

we have

$$\begin{aligned} d\Omega &= d\omega \wedge \omega - \omega \wedge d\omega \\ &= \Omega \wedge \omega - \omega \wedge \Omega \qquad \text{(Bianchy identity).} \end{aligned}$$

Therefore,

$$\begin{aligned} dP(\Omega) &= \operatorname{tr}(P'(\Omega) \wedge \Omega \wedge \omega) - \operatorname{tr}(P'(\Omega) \wedge \omega \wedge \Omega) \\ &= \operatorname{tr}(\Omega \wedge P'(\Omega) \wedge \omega) - \operatorname{tr}(P'(\Omega) \wedge \omega \wedge \Omega) \\ &= 0. \end{aligned}$$

□

Proposition 4.3.2. *Let P be an invariant homogeneous polynomial on $M_r(\mathbb{C})$ and let $\bigtriangledown_0$ and $\bigtriangledown_1$ be two $\mathbb{C}$-linear connections on E. Then, the closed differential forms $P(K_{\bigtriangledown_0})$ and $P(K_{\bigtriangledown_1})$ determine the same de Rham cohomology class.*

Proof. Consider the projection

$$p_B : \mathbb{R} \times B \to B$$

and the two connections $p_B^{-1}\bigtriangledown_0$, $p_B^{-1}\bigtriangledown_1$ on $p_B^{-1}E$. Denote $t : \mathbb{R} \times B \to \mathbb{R}$ the first projection and set

$$\bigtriangledown = (1-t)(p_B^{-1}\bigtriangledown_0) + t(p_B^{-1}\bigtriangledown_1).$$

Clearly, $\bigtriangledown$ is a connection on $p_B^{-1}E$. Denote

$$i_0 : B \to \mathbb{R} \times B, \qquad i_1 : B \to \mathbb{R} \times B$$

the two embeddings defined by setting

$$i_0(x) = (0, x), \qquad i_1(x) = (1, x).$$

Clearly, $i_0^{-1}p_B^{-1}(E) \simeq E$ and $i_1^{-1}p_B^{-1}(E) \simeq E$. Moreover, by construction $i_0^{-1}\bigtriangledown = \bigtriangledown_0$ and $i_1^{-1}\bigtriangledown = \bigtriangledown_1$. Therefore,

$$P(K_{\bigtriangledown_0}) = P(K_{i_0^{-1}\bigtriangledown}) = i_0^* P(K_{\bigtriangledown}),$$

$$P(K_{\bigtriangledown_1}) = P(K_{i_1^{-1}\bigtriangledown}) = i_1^* P(K_{\bigtriangledown}).$$

Using the isomorphism between de Rham cohomology and the usual cohomology together with the homotopy theorem, we see that the maps

$$i_0^* = \mathrm{H}^{\cdot}_{dR}(\mathbb{R} \times B; \mathbb{C}) \to \mathrm{H}^{\cdot}_{dR}(B; \mathbb{C})$$
$$i_1^* = \mathrm{H}^{\cdot}_{dR}(\mathbb{R} \times B; \mathbb{C}) \to \mathrm{H}^{\cdot}_{dR}(B; \mathbb{C})$$

are equal. The conclusion follows directly. □

Proposition 4.3.3. *There is at least one* $\mathbb{C}$*-linear connection on* E*.*

Proof. This is clear if E is trivializable. In general, we may find a locally finite covering $\mathcal{U}$ of B by open subsets U such that $E_{|U}$ is trivializable. For each $U \in \mathcal{U}$, fix a connection ∇_U on $E_{|U}$. Let $(\varphi_U)_{U \in \mathcal{U}}$ be a partition of unity subordinated to $\mathcal{U}$. Then,

$$\nabla = \sum_{u \in \mathcal{U}} \varphi_U \nabla_U$$

is a well-defined $\mathbb{C}$-linear connection on E. □

Corollary 4.3.4. *There is a canonical way to associate to any invariant polynomial on* $M_r(\mathbb{C})$ *a characteristic class for differentiable complex vector bundles of rank* r *with values in* $\mathbb{C}$*.*

Proof. Write $P = P_0 + P_1 + \cdots + P_k$ where each P_l is a homogeneous invariant polynomial of degree l. Let E be a differentiable complex vector bundle. By Proposition 4.3.3, there is at least one $\mathbb{C}$-linear connection ∇ on E. By Proposition 4.3.1, $P_l(K_\nabla)$ is a well-defined closed differential form of degree $2l$. Moreover, Proposition 4.3.2 shows that the cohomology class

$$\gamma_{P_l}(E) \in \mathrm{H}^{2l}(B; \mathbb{C})$$

of $P_l(K_\nabla)$ depends only on E. Set

$$\gamma_P(E) = \gamma_{P_0}(E) + \cdots + \gamma_{P_k}(E) \in \mathrm{H}^{\mathrm{ev}}(B; \mathbb{C}).$$

Since we have clearly

$$\gamma_P(f^{-1}E) = f^* \gamma_P(E)$$

for any differentiable map $f : B' \to B$, the conclusion follows. □

Proposition 4.3.5. *For any differentiable complex vector bundle* E *of rank* r *on* B*, we have*

$$c(E) = \gamma_{C_r}(E)$$

where C_r *is the invariant polynomial of* $A \in M_r(\mathbb{C})$ *defined by setting*

$$C_r(A) = \det(I + \frac{A}{2i\pi}).$$

Proof. (a) Assume first $r = 1$. Since B is paracompact and has finite dimension, it is possible to find a differentiable map

$$f : B \to \mathbb{P}_n(\mathbb{C})$$

such that $E \simeq f^{-1}U_n(\mathbb{C})$. It follows that we have only to treat the case of $U_n(\mathbb{C})$ on $\mathbb{P}_n(\mathbb{C})$. Moreover, since the restriction map

$$\mathrm{H}^1(\mathbb{P}_n(\mathbb{C}); \mathbb{C}) \to \mathrm{H}^1(\mathbb{P}_1(\mathbb{C}); \mathbb{C})$$

is injective, we can even assume $n = 1$. In this case, we have to show that if $\bigtriangledown$ is a connection on $U_1(\mathbb{C})$, then $c_1(U_n(\mathbb{C}))$ is represented by $\dfrac{[K_\bigtriangledown]}{2i\pi}$ or in other words that

$$\int_{\mathbb{P}_1(\mathbb{C})} K_\bigtriangledown = 2i\pi.$$

As usual, set

$$V_0 = \{[z_0, z_1] : z_0 \neq 0\}, \qquad V_1 = \{[z_0, z_1] : z_1 \neq 0\}$$

and define the coordinates $v_0 : V_0 \to \mathbb{C}$ and $v_1 : V_1 \to \mathbb{C}$ by setting

$$v_0([z_0, z_1]) = \frac{z_1}{z_0}, \qquad v_1([z_0, z_1]) = \frac{z_0}{z_1}.$$

We know that $U_{1|V_0}$ has a frame s_0 defined by setting

$$s_0([z_0, z_1]) = \left((1, \frac{z_1}{z_0}), [z_0, z_1]\right)$$

and that similarly, $U_{1|V_1}$ has a frame s_1 defined by setting

$$s_1([z_0, z_1]) = \left((\frac{z_0}{z_1}, 1), [z_0, z_1]\right).$$

A connection $\bigtriangledown$ on $U_1(\mathbb{C})$ is thus characterized by the differential forms $\omega_0 \in C^1_\infty(V_0; U_1(\mathbb{C}))$, $\omega_1 \in C^1_\infty(V_1; U_1(\mathbb{C}))$ defined by the relations

$$\bigtriangledown s_0 = \omega_0 s_0, \qquad \bigtriangledown s_1 = \omega_1 s_1.$$

Moreover, since

$$s_0 = v_0 s_1$$

on $V_0 \cap V_1$, we have

$$\bigtriangledown s_0 = (dv_0)s_1 + v_0 \bigtriangledown s_1$$

and hence

$$\omega_0 = \frac{(dv_0)}{v_0} + \omega_1.$$

Conversely, the two 1-forms ω_0, ω_1 are given and satisfy the preceding gluing condition, we can use them to construct a unique connection $\bigtriangledown$ on $U_1(\mathbb{C})$. Let φ be a differentiable function on $\mathbb{C}$ such that

$$\varphi = 0 \quad \text{on } D(0, \frac{1}{4}), \qquad \varphi = 1 \quad \text{on } \complement D(0, \frac{3}{4}).$$

Set

$$\omega_0 = \varphi(v_0)\frac{dv_0}{v_0}$$

on V_0 and

$$\omega_1 = \left(1 - \varphi(\frac{1}{v_1})\right)\frac{dv_1}{v_1}$$

on V_1. On $V_0 \cap V_1$, we have

$$v_1 = \frac{1}{v_0},$$

hence,

$$dv_1 = -\frac{1}{v_0^2}dv_0$$

and

$$\frac{dv_1}{v_1} = -\frac{dv_0}{v_0}.$$

It follows that

$$\omega_0 = \frac{dv_0}{v_0} + \omega_1$$

on $V_0 \cap V_1$ and that the forms ω_0, ω_1 define a connection $\bigtriangledown$ on $U_1(\mathbb{C})$. For this connection, we have

$$\begin{aligned}\Omega_0 &= d\omega_0 + \omega_0 \wedge \omega_0 \\ &= d\left(\varphi(v_0)\frac{dv_0}{v_0}\right)\end{aligned}$$

on V_0. Since Ω_0 has compact support in V_0, $K_\bigtriangledown$ corresponds to the form obtained by extending Ω_0 by 0 outside V_0. Therefore,

$$\begin{aligned}\int_{\mathbb{P}_1(\mathbb{C})} K_\bigtriangledown = \int_{V_0} \Omega_0 &= \int_{\mathbb{C}} d\left(\varphi(z)\frac{dz}{z}\right) \\ &= \int_{D(0,1)} d\left(\varphi(z)\frac{dz}{z}\right) \\ &= \int_{\partial D(0,1)} \varphi(z)\frac{dz}{z} \\ &= \int_{\partial D(0,1)} \frac{dz}{z} = 2i\pi\end{aligned}$$

and the conclusion follows.

(b) In the general case, it is easy to find a differentiable map

$$f : B' \to B$$

such that $f^{-1}E \simeq L_1 \oplus \cdots \oplus L_r$ with $L_1, \cdots, L_r$ of rank 1 and for which

$$f^* : \mathrm{H}^{\cdot}(B; \mathbb{C}) \to \mathrm{H}^{\cdot}(B'; \mathbb{C})$$

is injective. Let $\bigtriangledown_1, \cdots, \bigtriangledown_r$ be connections on $L_1, \cdots, L_r$ and denote $\bigtriangledown = \bigtriangledown_1 \oplus \cdots \oplus \bigtriangledown_r$ the associated direct sum connection on $f^{-1}E$. In the local frame $e_1, \cdots, e_r$ of $f^{-1}(E)$ corresponding to local frames $e_1, \cdots, e_r$ of $L_1, \cdots, L_r$, the curvature matrix Ω has the diagonal form

$$\begin{pmatrix} \Omega_1 & & \\ & \ddots & \\ & & \Omega_r \end{pmatrix}$$

where Ω_k is the curvature form of $\bigtriangledown_k$ with respect to e_k. Therefore,

$$\det(I + \frac{\Omega}{2i\pi}) = (I + \frac{\Omega_1}{2i\pi}) \cdots (I + \frac{\Omega_r}{2i\pi})$$

and

$$\gamma_{C_r}(f^{-1}(E)) = \gamma_{C_1}(L_1) \cdots \gamma_{C_1}(L_r).$$

It follows from (a) that

$$\gamma_{C_r}(f^{-1}(E)) = c(L_1) \cdots c(L_r) = c(f^{-1}(E)).$$

Hence

$$f^* \gamma_{C_r}(E) = f^* c(E)$$

and the conclusion follows. □

Corollary 4.3.6. *For any $A \in M_r(\mathbb{C})$, denote $\sigma_1(A), \cdots, \sigma_r(A)$ the complex numbers defined by setting*

$$\det(I + tA) = 1 + t\sigma_1(A) + \cdots + t^r \sigma_r(A).$$

Then, $\sigma_1(A), \cdots, \sigma_r(A)$ are invariant homogeneous polynomials of degree $1, \cdots, r$ and we have

$$\begin{aligned} \gamma_{\sigma_1}(E) &= 2i\pi c_1(E) \\ &\vdots \\ \gamma_{\sigma_r}(E) &= (2i\pi)^r c_r(E) \end{aligned}$$

for any differentiable complex vector bundle E of rank r on B.

Proof. This follows directly from the equality

$$C_r(A) = 1 + \frac{\sigma_1(A)}{2i\pi} + \cdots + \frac{\sigma_r(A)}{(2i\pi)^r}.$$

$\square$

Corollary 4.3.7. *There is a one to one correspondence between invariant polynomials on $M_r(\mathbb{C})$ and characteristic classes for complex vector bundles of rank r with coefficients in $\mathbb{C}$.*

Proof. We know that any characteristic class γ for complex vector bundles of rank r may be written in a unique way as a polynomial $Q(c_1, \cdots, c_r)$ in Chern classes. It follows that

$$\gamma = \gamma_{Q\left(\frac{\sigma_1}{2i\pi}, \cdots, \frac{\sigma_r}{(2i\pi)^r}\right)}.$$

Conversely, let P be an invariant polynomial on $M_r(\mathbb{C})$. We know that for any matrix $A \in M_r(\mathbb{C})$ there is $T \in \mathrm{GL}_r(\mathbb{C})$ such that

$$B = T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ * & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & \lambda_r \end{pmatrix}$$

For

$$S(\epsilon) = \begin{pmatrix} \epsilon & & & \\ & \epsilon^2 & & \\ & & \ddots & \\ & & & \epsilon^r \end{pmatrix}$$

the matrix

$$C(\epsilon) = S(\epsilon)BS(\epsilon)^{-1}$$

is such that

$$C_{jk}(\epsilon) = \epsilon^{j-k}B_{jk}.$$

It follows that

$$\lim_{\epsilon \longrightarrow 0} C_{jk}(\epsilon) = \lambda_k \delta_{jk}.$$

But since P is invariant, we have

$$P(A) = P(B) = P(C(\epsilon)) = \lim_{\epsilon \longrightarrow 0} P(C(\epsilon)) = P(\mathrm{diag}(\lambda_1, \cdots, \lambda_r)).$$

It follows that $P(A)$ is a symmetric polynomial in the eigenvalues $\lambda_1, \cdots, \lambda_r$ of A. Therefore, there is a unique polynomial $Q(S_1, \cdots, S_r)$ such that

$$P(A) = Q(\sigma_1(A), \cdots, \sigma_r(A)).$$

It follows that

$$\gamma_P = Q(2i\pi c_1, \cdots, (2i\pi)^r c_r).$$

Hence, the conclusion. □

4.4 Chern character

Definition 4.4.1. Let X be a topological space and let $\mathcal{F}$ be an abelian sheaf. Hereafter, we denote

$$\widehat{\mathrm{H}}^{\cdot}(X; \mathcal{F})$$

the completion of the topological abelian group obtained by endowing the graded abelian group

$$\mathrm{H}^{\cdot}(X; \mathcal{F})$$

with the topology for which

$$\bigoplus_{k \geq l} \mathrm{H}^k(X; \mathcal{F}) \quad (l \in \mathbb{N})$$

is a basis of neighborhoods of 0. Of course, forgetting the topologies, we have

$$\widehat{\mathrm{H}}^{\cdot}(X; \mathcal{F}) \simeq \prod_{k \in \mathbb{N}} \mathrm{H}^k(X; \mathcal{F}).$$

Remark 4.4.2. Consider a formal series

$$P(x_1, \cdots, x_r) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_r=0}^{\infty} a_{k_1 \cdots k_r} x_1^{k_1} \cdots x_r^{k_r}$$

with coefficients in a ring with unit A. Assume $P(x_1, \cdots, x_r)$ is symmetric (i.e. such that

$$P(x_{\mu_1}, \cdots, x_{\mu_r}) = P(x_1, \cdots, x_r)$$

for any permutation μ of $1, \cdots, r$). Then, the associated homogeneous polynomials

$$P_m(x_1, \cdots, x_r) = \sum_{|k|=m} a_k x^k \qquad (m \geq 0).$$

are also symmetric. As was explained in Remark 3.2.4 all these polynomials may be written in a unique way as

$$Q_m(S_{r,1}(x_1, \cdots, x_r), \cdots, S_{r,r}(x_1, \cdots, x_r)).$$

This allows us to associate to P and to any complex vector bundle E of rank r on X a class $c_P(E) \in \widehat{\mathrm{H}}^{\cdot}(X; A)$ by setting

$$c_P(E) = \sum_{m=0}^{\infty} Q_m(c_1(E), \cdots, c_r(E)).$$

Note that if $f : Y \to X$ is a continuous map, then it follows from the preceding construction that

$$c_P(f^{-1}E) = f^* c_P(E).$$

Definition 4.4.3. The class $\mathrm{ch}(E) \in \widehat{\mathrm{H}}^{\cdot}(X; \mathbb{Q})$ given by the construction above for $A = \mathbb{Q}$ and

$$P(x_1, \cdots, x_r) = e^{x_1} + \cdots + e^{x_r},$$

is called the *Chern character* of E.

Proposition 4.4.4. *Let X be a topological space. Assume E and F are complex vector bundles on X. Then,*

(a) $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$;

(b) $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \smile \mathrm{ch}(F)$.

Proof. This follows directly from the splitting principle and the definition of the Chern character. □

Proposition 4.4.5. *There is a unique extension*

$$\mathrm{ch} : \mathcal{K}^{\mathrm{b}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B)) \to \widehat{\mathrm{H}}^{\cdot}(B; \mathbb{Z})$$

of the usual Chern character

$$\mathrm{ch} : \mathcal{V}\mathrm{ect}_{\mathbb{C}}(B) \to \widehat{\mathrm{H}}^{\cdot}(B; \mathbb{Z})$$

which is invariant by isomorphism and such that

(i) for any $E^{\cdot}$ in $\mathcal{K}^{\mathrm{b}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$ we have

$$\mathrm{ch}(E^{\cdot}[1]) = -\,\mathrm{ch}(E^{\cdot})$$

(ii) for any distinguished triangle

$$E^{\cdot} \to F^{\cdot} \to G^{\cdot} \xrightarrow{+1}$$

of $\mathcal{K}^{\mathrm{b}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$, we have

$$\mathrm{ch}(F^{\cdot}) = \mathrm{ch}(E^{\cdot}) + \mathrm{ch}(G^{\cdot}).$$

This extension is given by the formula

$$\operatorname{ch}(E^\cdot) = \sum_{k\in\mathbb{Z}} (-1)^k \operatorname{ch}(E^k).$$

Proof. Uniqueness. Let $E^\cdot$ be an object of $\mathcal{K}^{\mathrm{b}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$ and let us prove that

$$\operatorname{ch}(E^\cdot) = \sum_{k\in\mathbb{Z}} (-1)^k \operatorname{ch}(E^k).$$

We know that there are integers a, b such that $E^k = 0$ if $k \notin [a,b]$. Let us proceed by increasing induction on $b-a$. If $b-a=0$, the result follows directly from (i) and the fact that ch extends the usual Chern character. Assume now that $b-a>0$. Remark that $E^\cdot$ is isomorphic to the mapping cone of

$$E^a[-a-1] \xrightarrow{d^a} \sigma^{>a} E^\cdot$$

where $\sigma^{>a}E^\cdot$ denotes the complex

$$0 \to E^{a+1} \to \cdots \to E^b \to 0$$

with E^{a+1} in degree $a+1$. It follows from (i) and (ii) that

$$\operatorname{ch}(\sigma^{>a}E^\cdot) = (-1)^{a-1}\operatorname{ch}(E^a) + \operatorname{ch}(E^\cdot).$$

Hence, by the induction hypothesis

$$\operatorname{ch}(E^\cdot) = (-1)^a \operatorname{ch}(E^a) + \sum_{k\geq a+1} (-1)^k \operatorname{ch}(E^k) = \sum_{k\in\mathbb{Z}} (-1)^k \operatorname{ch}(E^k).$$

Existence. Let us define $\operatorname{ch}(E^\cdot)$ as $\sum_{k\in\mathbb{Z}}(-1)^k \operatorname{ch}(E^k)$. Assume first that $E^\cdot \simeq 0$ in $\mathcal{K}^{\mathrm{b}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$. As is well-known, this means that $E^\cdot$ is split exact. Hence, if Z^k denotes the kernel of $d^k : E^k \to E^{k+1}$, we have

$$E^k \simeq Z^k \oplus Z^{k+1}$$

and

$$\operatorname{ch}(E^\cdot) = \sum_{k\in\mathbb{Z}} (-1)^k \operatorname{ch}(E^k) = \sum_{k\in\mathbb{Z}} (-1)^k \operatorname{ch}(Z^k) - \sum_{k\in\mathbb{Z}} (-1)^{k+1} \operatorname{ch}(Z^{k+1}) = 0.$$

Assume now that $M^\cdot(u^\cdot)$ is the mapping cone of

$$u^\cdot : E^\cdot \to F^\cdot.$$

By construction,

$$M^k(u^\cdot) = E^{k+1} \oplus F^k.$$

Hence,

$$\begin{aligned}\operatorname{ch}(M^\cdot(u^\cdot)) &= \sum_{k\in\mathbb{Z}}(-1)^k \left(\operatorname{ch}(E^{k+1}) + \operatorname{ch}(F^k)\right)\\ &= \operatorname{ch}(F^\cdot) - \operatorname{ch}(E^\cdot).\end{aligned}$$

Combining the two preceding results, we see that if

$$u^\cdot : E^\cdot \to F^\cdot$$

is an isomorphism in $\mathcal{K}^{\mathrm{b}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$, then $\operatorname{ch}(E^\cdot) = \operatorname{ch}(F^\cdot)$. It follows that ch is invariant by isomorphism and that (ii) is satisfied. ☐

Corollary 4.4.6. *The Chern character for complexes has the following properties:*

(i) if $E^\cdot$ and $F^\cdot$ are objects of $\mathcal{K}^{\mathrm{b}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$, then

$$\operatorname{ch}(E^\cdot \otimes F^\cdot) = \operatorname{ch}(E^\cdot) \smile \operatorname{ch}(F^\cdot);$$

(ii) if $b : B' \to B$ is a continuous map and $E^\cdot$ is an object of $\mathcal{K}^{\mathrm{b}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$, then

$$\operatorname{ch}(f^{-1}E^\cdot) = f^* \operatorname{ch}(E^\cdot).$$

Proof. This follows at once from the definition of ch for complexes and the similar properties of the usual Chern character. As a matter of fact, for (i), we have

$$(E^\cdot \otimes F^\cdot)^k = \bigoplus_{l\in\mathbb{Z}} E^l \otimes F^{k-l}$$

and

$$\begin{aligned}\operatorname{ch}(E^\cdot \otimes F^\cdot) &= \sum_{k\in\mathbb{Z}}(-1)^k \sum_{l\in\mathbb{Z}} \operatorname{ch}(E^l) \smile \operatorname{ch}(F^{k-l})\\ &= \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}(-1)^l \operatorname{ch}(E^l) \smile (-1)^{k-l}\operatorname{ch}(F^{k-l})\\ &= \operatorname{ch}(E^\cdot) \smile \operatorname{ch}(F^\cdot).\end{aligned}$$

As for (ii), we have

$$(f^{-1}E^\cdot)^k = f^{-1}(E^k)$$

and hence

$$\begin{aligned}\operatorname{ch}(f^{-1}E^\cdot) &= \sum_{k\in\mathbb{Z}}(-1)^k \operatorname{ch}(f^{-1}E^k)\\ &= \sum_{k\in\mathbb{Z}}(-1)^k f^* \operatorname{ch}(E^k)\\ &= f^* \operatorname{ch}(E^\cdot).\end{aligned}$$

□

Definition 4.4.7. The *Todd class* of a complex vector bundle E with base X is the class $\operatorname{td}(E) \in \widehat{\mathrm{H}}^{\cdot}(X;\mathbb{Q})$ given by the construction of Remark 4.4.2 for $A = \mathbb{Q}$ and

$$P(x_1, \cdots, x_r) = \frac{x_1}{1 - e^{-x_1}} \cdots \frac{x_r}{1 - e^{-x_r}}.$$

Remark 4.4.8. It follows directly from the preceding definition that

$$\operatorname{td}(E \oplus F) = \operatorname{td}(E) \smile \operatorname{td}(F)$$

if E and F are complex vector bundles on the same base.

Proposition 4.4.9. *Let E be a complex vector bundle of rank r on the topological space X. Then,*

$$\operatorname{ch}(\bigwedge^{\cdot} E) = c_r(E^*)/\operatorname{td}(E^*).$$

Proof. By the splitting principle, we may assume that $E = L_1 \oplus \cdots \oplus L_r$ where $L_1, \cdots, L_r$ are line bundles. Set $x_1 = c_1(L_1), \cdots, x_r = c_1(L_r)$. We have

$$\bigwedge^{\cdot} E = (\bigwedge^{\cdot} L_1) \otimes \cdots \otimes (\bigwedge^{\cdot} L_r).$$

Hence,

$$\operatorname{ch}(\bigwedge^{\cdot} E) = \operatorname{ch}(\bigwedge^{\cdot} L_1) \cdots \operatorname{ch}(\bigwedge^{\cdot} L_r) = (1 - e^{x_1}) \cdots (1 - e^{x_r}).$$

On the other hand,

$$c(E^*) = c(L_1^*) \cdots c(L_r^*) = (1 - x_1) \cdots (1 - x_r)$$

and

$$\operatorname{td}(E^*) = \operatorname{td}(L_1^*) \cdots \operatorname{td}(L_r^*) = \frac{-x_1}{1 - e^{x_1}} \cdots \frac{-x_r}{1 - e^{x_r}}.$$

It follows that

$$c_r(E^*) = (-x_1) \cdots (-x_r).$$

Hence,

$$c_r(E^*)/\operatorname{td}(E^*) = (1 - e^{x_1}) \cdots (1 - e^{x_r})$$

and the conclusion follows. □

4.5 Local chern character

Definition 4.5.1. Let $E^\cdot$ be an object of $\mathcal{K}^{\mathrm{b}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$ and let S be a closed subset of B. We say that $E^\cdot$ is *supported by* S if $E^\cdot_{\mathcal{B}\setminus S} \simeq 0$ in $\mathcal{K}^{\mathrm{b}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B \setminus S))$ and denote

$$\mathcal{K}^{\mathrm{b}}_S(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$$

the subcategory of $\mathcal{K}^{\mathrm{b}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$ formed by the complexes which are supported by S.

Assume the object $E^\cdot$ of $\mathcal{K}^{\mathrm{b}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$ is supported by S. Then,

$$\mathrm{ch}(E^\cdot)_{|B\setminus S} = \mathrm{ch}(E^\cdot_{|B\setminus S}) = 0$$

in $\widehat{\mathrm{H}}^\cdot(B \setminus S; \mathbb{Z})$. From the exact sequence

$$\widehat{\mathrm{H}}^\cdot_S(B;\mathbb{Z}) \xrightarrow{i} \widehat{\mathrm{H}}^\cdot(B;\mathbb{Z}) \xrightarrow{r} \widehat{\mathrm{H}}^\cdot(B \setminus S;\mathbb{Z})$$

it follows that $\mathrm{ch}(E^\cdot)$ is the image of a class in $\widehat{\mathrm{H}}^\cdot_S(B;\mathbb{Z})$. Although such a class is in general not unique, we shall prove the following result (cf [17]).

Proposition 4.5.2. *There is a unique way to define*

$$\mathrm{ch}_S : \mathcal{K}^{\mathrm{b}}_S(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B)) \to \widehat{\mathrm{H}}^\cdot_S(B;\mathbb{Z})$$

for any topological space B and any closed subset S of B in such a way that

(i) if $E^\cdot$ and $F^\cdot$ are isomorphic objects of $\mathcal{K}^{\mathrm{b}}_S(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$, then

$$\mathrm{ch}_S(E^\cdot) = \mathrm{ch}_S(F^\cdot);$$

(ii) for any $E^\cdot$ in $\mathcal{K}^{\mathrm{b}}_S(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$ we have

$$i(\mathrm{ch}_S(E^\cdot)) = \mathrm{ch}\, E^\cdot;$$

(iii) if $b : B' \to B$ is a continuous map and $E^\cdot$ is an object of $\mathcal{K}^{\mathrm{b}}_S(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$ then

$$\mathrm{ch}_{S'}(f^{-1}E^\cdot) = f^* \mathrm{ch}_S(E^\cdot)$$

for any closed subset S' of B' such that $f^{-1}(S) \subset S'$.

Moreover,

(iv) for any $E^\cdot \in \mathcal{K}^{\mathrm{b}}_S(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$ we have

$$\mathrm{ch}_S(E^\cdot[1]) = -\,\mathrm{ch}_S(E^\cdot);$$

(v) *for any distinguished triangle*

$$E^\cdot \to F^\cdot \to G^\cdot \xrightarrow{+1}$$

of $\mathcal{K}^{\mathrm{b}}_S(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$ *we have*

$$\mathrm{ch}_S(F^\cdot) = \mathrm{ch}_S(E^\cdot) + \mathrm{ch}_S(G^\cdot).$$

Proof. Uniqueness. By Lemma 4.5.3 below, we know that to any

$$E^\cdot \in \mathcal{K}^{\mathrm{b}}_S(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$$

is canonically associated a continuous map

$$s : B \to \tilde{B},$$

a closed subset $\tilde{S}$ and an object $\tilde{E}^\cdot$ of $\mathcal{K}^{\mathrm{b}}_{\tilde{S}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(\tilde{B}))$ such that

(i) $s^{-1}(\tilde{S}) \subset S$;

(ii) $s^{-1}\tilde{E}^\cdot \simeq E^\cdot$;

(iii) the map
$$\tilde{\imath} : \widehat{\mathrm{H}}^\cdot_{\tilde{S}}(\tilde{B};\mathbb{Z}) \to \widehat{\mathrm{H}}^\cdot(\tilde{B};\mathbb{Z})$$
is injective.

It follows that
$$\mathrm{ch}_S(E^\cdot) = \mathrm{ch}_S(s^{-1}\tilde{E}^\cdot) = s^*\,\mathrm{ch}_{\tilde{S}}(\tilde{E}^\cdot)$$
and since $\mathrm{ch}_{\tilde{S}}(\tilde{E}^\cdot)$ is the unique cohomology class such that

$$\tilde{\imath}(\mathrm{ch}_{\tilde{S}}(\tilde{E}^\cdot)) = \mathrm{ch}(\tilde{E}^\cdot),$$

we get the conclusion.

Existence. Using the notations introduced above, we define $\mathrm{ch}_S(E^\cdot)$ by setting
$$\mathrm{ch}_S(E^\cdot) = s^*\,\mathrm{ch}_{\tilde{S}}(\tilde{E}^\cdot)$$
where $\mathrm{ch}_{\tilde{S}}(\tilde{E}^\cdot)$ is characterized by the relation

$$\tilde{\imath}(\mathrm{ch}_{\tilde{S}}(\tilde{E}^\cdot)) = \mathrm{ch}(\tilde{E}^\cdot).$$

Note that from this definition it follows easily that $\mathrm{ch}_S(E^\cdot) = 0$ if $E^\cdot$ is exact. As a matter of fact, by construction of $\tilde{S}$ we have in this case $s(B) \cap \tilde{S} = \emptyset$.

Let $E^\cdot_1$, $E^\cdot_2$ be two objects of $\mathcal{K}^{\mathrm{b}}_{S_1}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B_1))$ and $\mathcal{K}^{\mathrm{b}}_{S_2}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B_2))$. Denote $\tilde{B}_1$, $\tilde{S}_1$, $\tilde{E}^\cdot_1$ and $\tilde{B}_2$, $\tilde{S}_2$, $\tilde{E}^\cdot_2$ the objects associated to $E^\cdot_1$ and $E^\cdot_2$ by Lemma 4.5.3 and let $s_1 : B_1 \to \tilde{B}_1$, $s_2 : B_2 \to \tilde{B}_2$ be the canonical maps.

(a) Assume first that $b : B_2 \to B_1$ is a continuous map, that $E_2^{\cdot} \simeq b^{-1}E_1^{\cdot}$ in $\mathcal{C}^{\mathrm{b}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B_2))$ and that $b^{-1}(S_1) \subset S_2$ and let us prove that

$$b^* \operatorname{ch}_{S_1}(E_1) = \operatorname{ch}_{S_2}(E_2^{\cdot}).$$

By construction of $\tilde{B}_2$ and $\tilde{B}_1$, we get a continuous map

$$\tilde{b} : \tilde{B}_2 \to \tilde{B}_1$$

such that $\tilde{b} \circ s_2 = s_1 \circ b$ and an isomorphism

$$\tilde{E}_2^{\cdot} \simeq \tilde{b}^{-1}\tilde{E}_1^{\cdot}$$

in $\mathcal{C}^{\mathrm{b}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(\tilde{B}_2))$. From the definitions of $\tilde{S}_1$ and $\tilde{S}_2$, it follows also that

$$\tilde{b}^{-1}\tilde{S}_1 \subset \tilde{S}_2.$$

Using Corollary 4.4.6, we see that

$$\operatorname{ch}(\tilde{E}_2^{\cdot}) = \tilde{b}^* \operatorname{ch}(\tilde{E}_1^{\cdot}).$$

This entails that

$$\begin{aligned}\tilde{i}_2 \operatorname{ch}_{\tilde{S}_2}(\tilde{E}_2^{\cdot}) &= \tilde{b}^*\tilde{i}_1 \operatorname{ch}_{\tilde{S}_1}(\tilde{E}_1^{\cdot}) \\ &= \tilde{i}_2(\tilde{b}^* \operatorname{ch}_{\tilde{S}_1}(\tilde{E}_1^{\cdot}))\end{aligned}$$

and hence that

$$\operatorname{ch}_{\tilde{S}_2}(\tilde{E}_2) = \tilde{b}^* \operatorname{ch}_{\tilde{S}_1}(\tilde{E}_1^*).$$

Therefore,

$$\begin{aligned}\operatorname{ch}_{S_2}(E_2) = s_2^* \operatorname{ch}_{\tilde{S}_2}(\tilde{E}_2^{\cdot}) &= s_2^*\tilde{b}^* \operatorname{ch}_{\tilde{S}_1}(\tilde{E}_1^{\cdot}) \\ = b^* s_1^* \operatorname{ch}_{\tilde{S}_1}(\tilde{E}_1^{\cdot}) &= b^* \operatorname{ch}_{S_1(E_1^{\cdot})}\end{aligned}$$

and the conclusion follows.

Let $E^{\cdot} \in \mathcal{K}^{\mathrm{b}}_S(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$. Applying the result obtained above for $B_1 = B_2$, $b = \mathrm{id}$, $E_1^{\cdot} = E^{\cdot}$, we see that $\operatorname{ch}_S(E^{\cdot})$ depends only on the isomorphy class of $E^{\cdot}$ in $\mathcal{C}^{\mathrm{b}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$. By a similar reasoning, we see also that

$$\operatorname{ch}_S(E^{\cdot}[1]) = -\operatorname{ch}_S(E^{\cdot}).$$

(b) Assume now that $B_1 = B_2$ and $S_1 = S_2$. Denoting B and S these spaces, let us prove that

$$\operatorname{ch}_S(E_1^{\cdot} \oplus E_2^{\cdot}) = \operatorname{ch}_S(E_1^{\cdot}) + \operatorname{ch}_S(E_2^{\cdot}).$$

Set $\tilde{C} = \tilde{B}_1 \times_B \tilde{B}_2$. Denote $t : B \to \tilde{C}$ the map deduced from from s_1 and s_2 and $p_1 : \tilde{C} \to \tilde{B}_1$, $p_2 : \tilde{C} \to \tilde{B}_2$ the two projections. Set $\tilde{F}_1^{\cdot} = p_1^{-1}\tilde{E}_1^{\cdot}$, $\tilde{F}_2^{\cdot} = p_2^{-1}\tilde{E}_2^{\cdot}$ and $T = p_1^{-1}(\tilde{S}_1) \cup p_2^{-1}(\tilde{S}_2)$. Thanks to Lemma 4.5.5, we know that

$$i : \widehat{\mathrm{H}}_T^{\cdot}(\tilde{C}; \mathbb{Z}) \to \mathrm{H}^{\cdot}(\tilde{C}; \mathbb{Z})$$

is injective. Since

$$\begin{aligned} i(\mathrm{ch}_T(\tilde{F}_1^{\cdot} \oplus \tilde{F}_2^{\cdot})) &= \mathrm{ch}(\tilde{F}_1^{\cdot} \oplus \tilde{F}_2^{\cdot}) \\ &= \mathrm{ch}(\tilde{F}_1^{\cdot}) + \mathrm{ch}(\tilde{F}_2^{\cdot}) \\ &= i(\mathrm{ch}_T(\tilde{F}_1^{\cdot}) + \mathrm{ch}_T(\tilde{F}_2^{\cdot})) \end{aligned}$$

it follows that

$$\mathrm{ch}_T(\tilde{F}_1^{\cdot} \oplus \tilde{F}_2^{\cdot}) = \mathrm{ch}_T(\tilde{F}_1^{\cdot}) + \mathrm{ch}_T(\tilde{F}_2^{\cdot}).$$

Using the fact that $t^{-1}(T) = s_1^{-1}(\tilde{S}_1) \cap s_2^{-1}(\tilde{S}_2) \subset S$ and the isomorphisms

$$t^{-1}\tilde{F}_1^{\cdot} \simeq s_1^{-1}\tilde{E}_1^{\cdot} \simeq E_1^{\cdot}, \qquad t^{-1}\tilde{F}_2^{\cdot} \simeq s_2^{-1}\tilde{E}_2^{\cdot} \simeq E_2^{\cdot},$$

we get that

$$\begin{aligned} \mathrm{ch}_S(\tilde{E}_1^{\cdot} \oplus \tilde{E}_2^{\cdot}) &= t^* \mathrm{ch}_T(\tilde{F}_1^{\cdot} \oplus \tilde{F}_2^{\cdot}) \\ &= t^* \mathrm{ch}_T(\tilde{F}_1^{\cdot}) + t^* \mathrm{ch}_T(\tilde{F}_2^{\cdot}) \\ &= \mathrm{ch}_S(\tilde{E}_1^{\cdot}) + \mathrm{ch}_S(\tilde{E}_2^{\cdot}). \end{aligned}$$

(c) With the same notations as in (b), let us now prove that if $M^{\cdot}(u^{\cdot})$ is the mapping cone of the morphism

$$u^{\cdot} : E_1^{\cdot} \to E_2^{\cdot}$$

of $\mathcal{C}_S^{\mathrm{b}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$ then $\mathrm{ch}_S(M^{\cdot}(u^{\cdot})) = \mathrm{ch}_S(E_2^{\cdot}) - \mathrm{ch}_S(E_1^{\cdot})$. To this end, consider the complex vector bundle $p : H \to B$ whose fiber at $b \in B$ is

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{b}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}})}(E_b^{\cdot}, F_b^{\cdot}),$$

a continuous local frame being obtained by fixing continuous local frames for each E^k and each F^k and associating to a morphism $v^{\cdot}$ the components of the matrices of the v^k's with respect to the fixed frames. Set $F_1^{\cdot} = p^{-1}E_1^{\cdot}$ and $F_2^{\cdot} = p^{-1}E_2^{\cdot}$. By construction, there is a canonical morphism

$$v^{\cdot} : F_1^{\cdot} \to F_2^{\cdot}$$

and we can associate to $u^{\cdot} : E_1^{\cdot} \to E_2^{\cdot}$ a section

$$\sigma_{u^{\cdot}} : B \to H$$

making the diagram

$$\begin{tikzcd} \sigma_{u^\cdot}^{-1}(F_1^\cdot) \arrow[r, "\sigma_{u^\cdot}^{-1}(v^\cdot)"] \arrow[d, "\wr"] & \sigma_{u^\cdot}^{-1}(F_2^\cdot) \arrow[d, "\wr"] \\ E_1^\cdot \arrow[r, "u^\cdot"'] & E_2^\cdot \end{tikzcd}$$

commutative. It follows that

$$\sigma_{u^\cdot}^{-1} M^\cdot(v^\cdot) \simeq M^\cdot(u^\cdot)$$

in $\mathcal{C}^{\mathrm{b}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$ and hence that

$$\mathrm{ch}_S(M^\cdot(u^\cdot)) = \sigma_{u^\cdot}^* \,\mathrm{ch}_S(M^\cdot(v^\cdot)).$$

Since $\sigma_{u^\cdot}$ and $\sigma_{0^\cdot}$ are clearly homotopic, we see that

$$\begin{aligned} \mathrm{ch}_S(M^\cdot(u^\cdot)) &= \mathrm{ch}_S(M^\cdot(0^\cdot)) \\ &= \mathrm{ch}_S(E_1^\cdot[-1] \oplus E_2^\cdot) \\ &= \mathrm{ch}_S(E_2^\cdot) - \mathrm{ch}_S(E_1^\cdot). \end{aligned}$$

(d) When $u^\cdot$ is an isomorphism in $\mathcal{K}^{\mathrm{b}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$, $M^\cdot(u^\cdot)$ is exact and we get

$$\mathrm{ch}_S(E_2^\cdot) = \mathrm{ch}_S(E_1^\cdot).$$

It follows that for any distinguished triangle

$$E_1^\cdot \to E_2^\cdot \to E_3^\cdot \xrightarrow{+1}$$

of $\mathcal{K}^{\mathrm{b}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$, we have

$$\mathrm{ch}_S(E_3^\cdot) = \mathrm{ch}_S(E_2^\cdot) - \mathrm{ch}_S(E_1^\cdot)$$

and this completes the proof. □

Lemma 4.5.3. *Let S be a closed subset of the topological space B. Then, to any object $E^\cdot$ of $\mathcal{C}^{\mathrm{b}}_S(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$ one can associate canonically*

(i) a continuous projection $p : \tilde{B} \to B$ with a canonical section $s : B \to \tilde{B}$;

(ii) a closed subset $\tilde{S}$ of $\tilde{B}$ such that $s^{-1}(\tilde{S}) \subset S$ and for which the canonical map

$$\tilde{\imath} : \widehat{\mathrm{H}}^\cdot_{\tilde{S}}(\tilde{B}; \mathbb{Q}) \to \mathrm{H}^\cdot(\tilde{B}; \mathbb{Q})$$

is injective;

(iii) an object $\tilde{E}^{\cdot}$ *of* $\mathcal{C}^{\mathrm{b}}_{\tilde{S}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(\tilde{B}))$ *such that*

$$s^{-1}(\tilde{E}^{\cdot}) \simeq E^{\cdot}.$$

Proof. The case $S = B$ being obvious, we shall assume $S \neq B$.

(a) Construction of $\tilde{B}$. Let $\nu. = (\nu_l)_{l\in\mathbb{Z}}$ be a family of natural integers for which $\{l \in \mathbb{Z} : \nu_l \neq 0\}$ is finite and let V be a finite dimensional complex vector space. By a flag of nationality $\nu.$ of V, we mean an increasing family $V. = (V_l)_{l\in\mathbb{Z}}$ of complex vector subspaces of V such that

$$\dim V_{l+1} = \dim V_l + \nu_{l+1}.$$

Generalizing what has been done for complex Grassmannians, it is easy to see that the set $\mathrm{Fl}_{\nu.}(V)$ of flags of nationality $\nu.$ of V has a canonical structure of differential manifold. Working as in the construction of the projective bundle associated to a complex vector bundle, we see more generally that any complex vector bundle $E \to B$ gives rise to the flag bundle $\mathrm{Fl}_{\nu.}(E) \to B$.

In the special case where

$$E = \oplus_{k\in\mathbb{Z}} E^k$$

and

$$\nu. = (\mathrm{rk}\, E^l)_{l\in\mathbb{Z}}$$

the flag bundle $\mathrm{Fl}_{\nu.}(E) \to B$ has a canonical section given by

$$b \mapsto E_{l,b}$$

where

$$E_l = \oplus_{k\leq l} E^k$$

for any $l \in \mathbb{Z}$. Let us define $\tilde{B}$ as the closed subset of $\mathrm{Fl}_{\nu.}(E)$ formed by flags $F.$ such that

$$E_{l-1,b} \subset F_l \subset E_{l+1,b}$$

for any $l \in \mathbb{Z}$ if $F. \in \mathrm{Fl}_{\nu.}(E^{\cdot})_b$. We denote $p : \tilde{B} \to B$ the map induced by the canonical projection $\mathrm{Fl}_{\nu.}(E) \to B$.

(b) Construction of $\tilde{E}^{\cdot}$ and $\tilde{S}$. Denote $\tilde{F}_l$ the complex vector bundle on $\tilde{B}$ whose fiber at $F. \in \tilde{B}$ is F_l. By construction,

$$\tilde{F}_l \subset \tilde{F}_{l+1}, \qquad p^{-1}E_l \subset p^{-1}E_{l+1}, \qquad p^{-1}E_{l-1} \subset \tilde{F}_l \subset p^{-1}E_{l+1}$$

for any $l \in \mathbb{Z}$. Set

$$\tilde{E}^k = \tilde{F}_k / p^{-1}E_{k-1}.$$

Clearly, $\tilde{E}^k$ is a complex vector bundle of rank ν_k on $\tilde{B}$. Denote

$$\tilde{d}^k : \tilde{E}^k \to \tilde{E}^{k+1}$$

the morphism of complex vector bundles deduced from the inclusions $\tilde{F}_k \subset \tilde{F}_{k+1}$, $p^{-1}E_{k-1} \subset p^{-1}E_k$. Since

$$\tilde{F}_k \subset p^{-1}E_{k+1}$$

we have $\tilde{d}^{k+1} \circ \tilde{d}^k = 0$ and $\tilde{E}^\cdot = (\tilde{E}^k, \tilde{d}^k)_{k\in\mathbb{Z}}$ is a complex of complex vector bundles on $\tilde{B}$. We denote $\tilde{S}$ the support of $\tilde{E}^\cdot$ in $\tilde{B}$.

(c) Construction of s. Let us define $s : B \to \tilde{B}$ as the map which associates to any $b \in B$ the flag $F. \in \mathrm{Fl}_{\nu.}(E)_b$ defined by setting

$$F_l = \{x \in \bigoplus_{k\leq l+1} E^k_b : x_{l+1} = d^l x_l\}.$$

One checks easily that s is continuous and that

$$s^{-1}\tilde{E}^\cdot \simeq E^\cdot$$

in $\mathcal{C}^{\mathrm{b}}(\mathcal{V}\mathrm{ect}_{\mathbb{C}}(B))$. In particular, we see that the support of $E^\cdot$ is $s^{-1}(\tilde{S})$ and hence that $s^{-1}(\tilde{S}) \subset S$.

(d) Construction of the sequence $(\lambda_k)_{k\in\mathbb{Z}}$ if $\tilde{S} \neq \tilde{B}$. Since $\tilde{S} \neq \tilde{B}$, the complex $\tilde{E}$ is exact at some point $\tilde{b}$ of $\tilde{B}$. It follows that there is a unique sequence $\lambda. = (\lambda_k)_{k\in\mathbb{Z}}$ of natural numbers such that

$$\nu_k = \lambda_k + \lambda_{k+1}.$$

As a matter of fact, if $\tilde{Z}^k_{\tilde{b}}$ is the kernel of $\tilde{d}^k_{\tilde{b}} : \tilde{E}^k_{\tilde{b}} \to \tilde{E}^{k+1}_{\tilde{b}}$, we have

$$\tilde{E}^k_{\tilde{b}} \simeq \tilde{Z}^k_{\tilde{b}} \oplus \tilde{Z}^{k+1}_{\tilde{b}}$$

and we may take $\lambda_k = \dim \tilde{Z}^k_{\tilde{b}}$. The existence follows. As for the uniqueness, it is sufficient to note that

$$\lambda_l = \sum_{k<l} (-1)^{k-l+1} \nu_k.$$

(e) Characterization of $\tilde{B} \setminus \tilde{S}$ if $\tilde{S} \neq \tilde{B}$. Let us prove that a flag $F. \in \tilde{B}$ above $b \in B$ is not in $\tilde{S}$ if and only if

$$\dim(F_k \cap E_{k,b}/E_{k-1,b}) = \lambda_k \tag{*}$$

for all $k \in \mathbb{Z}$. To this end, recall that the fiber of $\tilde{E}^\cdot$ at $F.$ is isomorphic to the complex

$$\cdots \to F_{k-1}/E_{k-2,b} \to F_k/E_{k-1,b} \to F_{k+1}/E_{k,b} \to \cdots$$

the k-cocycles of which are given by

$$\tilde{Z}^k_{F.} = F_k \cap E_{k,b}/E_{k-1,b}.$$

Hence, if $\tilde{E}^{\cdot}$ is exact at $F.$, (*) follows from the construction of $\lambda.$ in (a). Conversely, assume (*) holds for all $k \in \mathbb{Z}$ and denote $\tilde{B}^k_{F.}$ the k-coboundaries of $\tilde{E}^{\cdot}_{F.}$. The exact sequence

$$0 \to \tilde{Z}^{k-1}_{F.} \to \tilde{E}^{k-1}_{F.} \to \tilde{B}^k_{F.} \to 0$$

shows that

$$\dim \tilde{B}^k_{F.} = \dim \tilde{E}^{k-1}_{F.} - \dim \tilde{Z}^{k-1}_{F.} = \nu_{k-1} - \lambda_{k-1} = \lambda_k.$$

It follows that $\dim \tilde{B}^k_{F.} = \dim \tilde{Z}^k_{F.}$ and hence that $\tilde{B}^k_{F.} = \tilde{Z}^k_{F.}$.

(f) Cohomology of $\tilde{B} \setminus \tilde{S}$ if $\tilde{S} \neq \tilde{B}$. Recall that if E is an n-dimensional complex vector bundle on B and l is an integer such that $0 \leq l \leq n$, Then $G_l(E)$ denotes the Grassmannian bundle whose fiber at $b \in B$ is the set of complex vector subspaces of dimension l of E_b. Denote G the fibered product of the Grassmannian bundles $G_{\lambda_k}(E^k)$ $(k \in \mathbb{Z})$. For any flag $F.$ of $\tilde{B} \setminus \tilde{S}$ above b and any $k \in \mathbb{Z}$, part (e) shows that

$$(F_k \cap E_{k,b})/E_{k-1,b}$$

is λ_k-dimensional. Since

$$E_{k,b}/E_{k-1,b} \simeq E^k$$

this gives us a continuous map

$$f : \tilde{B} \setminus \tilde{S} \to G.$$

We shall prove that this map is a fiber bundle whose fibers are isomorphic to $\mathbb{C}^d$ with $d = \sum_{k \in \mathbb{Z}} \lambda_k^2$. This will entail that

$$f^* : \mathrm{H}^{\cdot}(G; \mathbb{Q}) \to \mathrm{H}^{\cdot}(\tilde{B} \setminus \tilde{S}; \mathbb{Q})$$

is an isomorphism.

For any $k \in \mathbb{Z}$, let L_k be an element of $G_{\lambda_k}(E^k)$ above $b \in B$. Denote $L.$ the element of G associated to the family $(L_k)_{k \in \mathbb{Z}}$. A flag $F.$ of $\tilde{B}$ above b belongs to $f^{-1}(L.)$ if and only if the image of

$$F_k \cap E_{k,b}/E_{k-1,b}$$

in E^k is equal to L_k for any $k \in \mathbb{Z}$. Assume F_l is known for $l \geq k+1$, then F_k may be chosen arbitrarily between the subspaces of dimension $\sum_{l\leq k} \nu_l$ of $F_{k+1} \cap E_{k+1,b}$ such that

$$F_k \cap E_{k,b} = E_{k-1,b} \oplus L_k.$$

These subspaces are in bijection with the subspaces G_k of dimension $\nu_k - \lambda_k = \lambda_{k+1}$ of

$$F_{k+1} \cap E_{k+1,b}/E_{k-1,b} \oplus L_k$$

such that $G_k \cap E_{k,b}/E_{k-1,b} \oplus L_k = 0$. Since

$$\dim(F_{k+1} \cap E_{k+1,b}/E_{k+1,b}) = \lambda_{k+1}$$

and

$$\dim(E_{k,b}/E_{k-1,b} \oplus L_k) = \nu_k - \lambda_k = \lambda_{k+1},$$

we have

$$\dim(F_{k+1} \cap E_{k+1,b}/E_{k-1,b} \oplus L_k) = \lambda_{k+1} + \nu_k - \lambda_k = 2\lambda_{k+1}.$$

It follows that G_k may be chosen arbitrarily among the supplementary subspaces of a fixed λ_{k+1} dimensional subspace in a space of dimension $2\lambda_{k+1}$. Thanks to Lemma 4.5.4, these subspaces are in bijection with $\mathbb{C}^{\lambda_{k+1}^2}$. The conclusion follows by decreasing induction on k.

Now, consider the continuous map

$$g : G \to \tilde{B} \setminus \tilde{S}$$

defined by sending an element $L_\cdot$ of G above $b \in B$ to the flag

$$(E_{l-1,b} \oplus L_l \oplus L_{l+1})_{l\in\mathbb{Z}}.$$

By construction,

$$f \circ g = \mathrm{id}_G\,.$$

Therefore,

$$g^* : \mathrm{H}^\cdot(\tilde{B} \setminus \tilde{S}; \mathbb{Q}) \to \mathrm{H}^\cdot(G; \mathbb{Q})$$

is the inverse of

$$f^* : \mathrm{H}^\cdot(G; \mathbb{Q}) \to \mathrm{H}^\cdot(\tilde{B} \setminus \tilde{S}; \mathbb{Q}).$$

(g) Injectivity of $\tilde{i}$. A long exact sequence of cohomology shows that the canonical morphism

$$\tilde{i} : \mathrm{H}^\cdot_{\tilde{S}}(\tilde{B}; \mathbb{Q}) \to \mathrm{H}^\cdot(\tilde{B}; \mathbb{Q})$$

is injective, if the canonical morphism

$$\tilde{j}^* : \mathrm{H}^{\cdot}(\tilde{B}; \mathbb{Q}) \to \mathrm{H}^{\cdot}(\tilde{B} \setminus \tilde{S}; \mathbb{Q})$$

associated to the inclusion $\tilde{j} : \tilde{B} \setminus \tilde{S} \to \tilde{B}$ is surjective. Set

$$G' = \prod_{k \in \mathbb{Z}} G_{\nu_k}(E^k \oplus E^{k+1})$$

and denote

$$f' : \tilde{B} \to G'$$

the continuous map which sends a flag $F.$ of $\tilde{B}_b$ to the element

$$(F_k / E_{k-1,b})_{k \in \mathbb{Z}}$$

of G'_b. Clearly, $f' \circ \tilde{j} \circ g$ is the map

$$h : G \to G'$$

which sends an element $L. \in G_b$ to the element

$$(L_k \oplus L_{k+1})_{k \in \mathbb{Z}}$$

of G'_b. Since

$$h^* = g^* \circ (\tilde{j})^* \circ (f')^*$$

and since g^* is an isomorphism, it is sufficient to show that h^* is surjective. Denote

$$p_k : G \to G_{\lambda_k}(E^k)$$

the canonical projection and T_k the tautological bundle on $G_{\lambda_k}(E^k)$. Repeated applications of the Leray-Hirsch theorem show that

$$\mathrm{H}^{\cdot}(G; \mathbb{Q})$$

is a free $\mathrm{H}^{\cdot}(B; \mathbb{Q})$-module generated by the $p_k^* c_l(T_k)$. Denote

$$p'_k : G' \to G_{\nu_k}(E^k \oplus E^{k+1})$$

the canonical projection and T'_k the tautological bundle on $G_{\nu_k}(E^k \oplus E^{k+1})$. Clearly, we have

$$(p'_k \circ h)^{-1}(T'_k) \simeq p_k^{-1} T_k \oplus p_{k+1}^{-1} T_{k+1}.$$

Therefore,

$$(h^* \circ (p'_k)^*)(c.(T'_k)) = p_k^* c.(T_k) \smile p_{k+1}^* c.(T_{k+1}).$$

It follows that

$$p_k^* c^{\cdot}(T_k) \smile p_{k+1}^* c^{\cdot}(T_{k+1})$$

and its inverse both belong to the image of h^*. Hence, proceeding by decreasing induction on k, we see that $p_k^* c^{\cdot}(T_k) \in \operatorname{Im} h^*$ for all $k \in \mathbb{Z}$. This entails that $\operatorname{Im} h^* = \mathrm{H}^{\cdot}(G;\mathbb{Q})$ and the conclusion follows. $\square$

Lemma 4.5.4. *Let E, F be two finite dimensional complex vector spaces and set $G = E \oplus F$. Then, the vector subspace H of G such that $E \oplus H = G$ are canonically in bijection with the complex vector space*

$$\operatorname{Hom}(F, E).$$

whose dimension is $\dim E \cdot \dim F$.

Proof. Let $p_E : G \to E$ and $p_F : G \to F$ be the two projections associated to the decomposition $G = E \oplus F$. For any vector subspace H of G such that $E \oplus H = G$, the linear map

$$p_{F|H} : H \to F$$

is bijective. Denote q its inverse and set $h_H = p_E \circ q$. By construction,

$$q(x) = (h_H(x), x) \qquad \text{and} \qquad H = \{(h_H(x), x) : x \in F\}.$$

Now, let $h \in \operatorname{Hom}(F, E)$ and set

$$H_h = \{(h(x), x) : x \in F\}.$$

Clearly, $E \oplus H_h = G$ and one checks easily that the maps $H \mapsto h_H$ and $h \mapsto H_h$ are reciprocal bijections. $\square$

Lemma 4.5.5. *Let S_1, S_2 be two closed subsets of B and let $E_1^{\cdot}$, $E_2^{\cdot}$ be two bounded complexes supported respectively by S_1 and S_2. Denote*

$$q_1 : \tilde{B}_1 \to B, \quad s_1 : B \to \tilde{B}_1, \quad \tilde{S}_1, \quad \tilde{E}_1^{\cdot}$$

and

$$q_2 : \tilde{B}_2 \to B, \quad s_2 : B \to \tilde{B}_2, \quad \tilde{S}_2, \quad \tilde{E}_2^{\cdot}$$

the objects associated to $E_1^{\cdot}$ and $E_2^{\cdot}$ by Lemma 4.5.3. Set $\tilde{C} = \tilde{B}_1 \times_B \tilde{B}_2$ and denote $p_1 : \tilde{C} \to \tilde{B}_1$, $p_2 : \tilde{C} \to \tilde{B}_2$ the two projections. Then the canonical morphism

$$\mathrm{H}_T^{\cdot}(\tilde{C};\mathbb{Q}) \to \mathrm{H}^{\cdot}(\tilde{C};\mathbb{Q})$$

is injective if T is equal to one of the closed subsets

$$p_1^{-1}(\tilde{S}_1), \quad p_2^{-1}(\tilde{S}_2), \quad p_1^{-1}(\tilde{S}_1) \cap p_2^{-1}(\tilde{S}_2), \quad p_1^{-1}(\tilde{S}_1) \cup p_2^{-1}(\tilde{S}_2).$$

Proof. Thanks to Lemma 4.5.6, we have the following (horizontal) exact sequence of (vertical) complexes

$$\begin{tikzcd}
& 0 & 0 & 0 & \\
0 \arrow[r] & \mathrm{H}^{\cdot}_{\tilde{S}_1 \times (\tilde{B}_2 \setminus \tilde{S}_2)}(\tilde{B}_1 \times \tilde{B}_2 \setminus \tilde{S}_2;\mathbb{Q}) \arrow[r] \arrow[u] & \mathrm{H}^{\cdot}(\tilde{B}_1 \times \tilde{B}_2 \setminus \tilde{S}_2;\mathbb{Q}) \arrow[r, "\delta"] \arrow[u] & \mathrm{H}^{\cdot}(\tilde{B}_1 \setminus \tilde{S}_1 \times \tilde{B}_2 \setminus \tilde{S}_2;\mathbb{Q}) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathrm{H}^{\cdot}_{\tilde{S}_1 \times \tilde{B}_2}(\tilde{B}_1 \times \tilde{B}_2;\mathbb{Q}) \arrow[r, "\gamma"] \arrow[u, "\alpha"'] & \mathrm{H}^{\cdot}(\tilde{B}_1 \times \tilde{B}_2;\mathbb{Q}) \arrow[r] \arrow[u, "\beta"'] & \mathrm{H}^{\cdot}(\tilde{B}_1 \setminus \tilde{S}_1 \times \tilde{B}_2;\mathbb{Q}) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathrm{H}^{\cdot}_{\tilde{S}_1}(\tilde{B}_1;\mathbb{Q}) \otimes \mathrm{H}^{\cdot}_{\tilde{S}_2}(\tilde{B}_2;\mathbb{Q}) \arrow[r] \arrow[u] & \mathrm{H}^{\cdot}(\tilde{B}_1;\mathbb{Q}) \otimes \mathrm{H}^{\cdot}_{\tilde{S}_2}(\tilde{B}_2;\mathbb{Q}) \arrow[r] \arrow[u] & \mathrm{H}^{\cdot}(\tilde{B}_1 \setminus \tilde{S}_1;\mathbb{Q}) \otimes \mathrm{H}^{\cdot}_{\tilde{S}_2}(\tilde{B}_2;\mathbb{Q}) \arrow[r] \arrow[u] & 0 \\
& 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where cartesian products (resp. tensor products) are to be understood over B (resp. $\mathrm{H}^{\cdot}(B;\mathbb{Q})$). The third (vertical) complex being exact, the snake's Lemma combined with the fact that β is surjective shows that α is also surjective. Combining this result with the long exact sequence associated with the distinguished triangle

$$\mathrm{R}\Gamma_{\tilde{S}_1 \times \tilde{S}_2}(\tilde{B}_1 \times \tilde{B}_2;\mathbb{Q}) \to \mathrm{R}\Gamma_{\tilde{S}_1 \times \tilde{B}_2}(\tilde{B}_1 \times \tilde{B}_2;\mathbb{Q}) \to \mathrm{R}\Gamma_{\tilde{S}_1 \times \tilde{B}_2 \setminus \tilde{S}_2}(\tilde{B}_1 \times \tilde{B}_2 \setminus \tilde{S}_2;\mathbb{Q}) \xrightarrow{+1}$$

we get that

$$\mathrm{H}^{\cdot}_{\tilde{S}_1 \times \tilde{S}_2}(\tilde{B}_1 \times \tilde{B}_2;\mathbb{Q}) \to \mathrm{H}^{\cdot}_{\tilde{S}_1 \times \tilde{B}_2}(\tilde{B}_1 \times \tilde{B}_2;\mathbb{Q})$$

is injective. Since γ is also injective, we have established the result for $T = \tilde{S}_1 \times \tilde{S}_2$ and $T = \tilde{S}_1 \times \tilde{B}_2$. Since β and $\delta \circ \beta$ are both surjective we also get the result for $T = \tilde{B}_1 \times \tilde{S}_2$ and $T = (\tilde{S}_1 \times \tilde{B}_2) \cup (\tilde{B}_1 \times \tilde{S}_2)$. □

Lemma 4.5.6. *Let $E^{\cdot}$ be a bounded complex supported by the closed subset S of B. Let $q : \tilde{B} \to B$ and $\tilde{S}$ be as in Lemma 4.5.3. Then,*

(i) $\mathrm{H}^{\cdot}(\tilde{B} \setminus \tilde{S};\mathbb{Q})$ *is a finite free* $\mathrm{H}^{\cdot}(B;\mathbb{Q})$*-module;*

(ii) for any continuous map $b : C \to B$, we have the canonical morphism

$$\mathrm{H}^{\cdot}(\tilde{B} \setminus \tilde{S};\mathbb{Q}) \otimes_{\mathrm{H}^{\cdot}(B;\mathbb{Q})} \mathrm{H}^{\cdot}(C;\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^{\cdot}(\tilde{B} \setminus \tilde{S} \times_B C;\mathbb{Q}).$$

Proof. (i) Using the notions introduced in Lemma 4.5.3, we have

$$\mathrm{H}^{\cdot}(\tilde{B} \setminus \tilde{S};\mathbb{Q}) \simeq \mathrm{H}^{\cdot}(G;\mathbb{Q})$$

and the conclusion follows from Leray-Hirsch theorem.

(ii) Working as in part (c) of this same lemma, we see that

$$f \times_B \mathrm{id}_C : \tilde{B} \setminus \tilde{S} \times_B C \to G \times_B C$$

induces an isomorphism

$$\mathrm{H}^{\cdot}(G \times C; \mathbb{Q}) \simeq \mathrm{H}^{\cdot}(\tilde{B} \setminus \tilde{S} \times_B C; \mathbb{Q})$$

and the conclusion follows by applying once more Leray-Hirsch theorem. □

Proposition 4.5.7. *If $E_1^{\cdot}$ and $E_2^{\cdot}$ are complexes of complex vector bundles on B with support respectively in S_1 and S_2 then $E_1^{\cdot} \otimes E_2^{\cdot}$ has a support in $S_1 \cap S_2$ and*

$$\mathrm{ch}_{S_1 \cap S_2}(E_1^{\cdot} \otimes E_2^{\cdot}) = \mathrm{ch}_{S_1}(E_1^{\cdot}) \smile \mathrm{ch}_{S_2}(E_2^{\cdot}).$$

Proof. Let us use the notations of Lemma 4.5.5. Denote $t : B \to \tilde{C}$ the map associated to the canonical maps $s_1 : B \to \tilde{B}_1$, $s_2 : B \to \tilde{B}_2$ and set

$$\tilde{F}_1^{\cdot} = p_1^{-1}\tilde{E}_1^{\cdot}, \quad \tilde{F}_2^{\cdot} = p_2^{-1}\tilde{E}_2^{\cdot}, \quad \tilde{T}_1 = p_1^{-1}\tilde{S}_1, \quad \tilde{T}_2 = p_2^{-1}\tilde{S}_2.$$

Clearly, $\tilde{F}_1^{\cdot}$ and $\tilde{F}_2^{\cdot}$ have support respectively in $\tilde{T}_1$ and $\tilde{T}_2$ and $\tilde{F}_1^{\cdot} \otimes \tilde{F}_2^{\cdot}$ has support in $\tilde{T}_1 \cap \tilde{T}_2$. Since

$$E_1^{\cdot} = t^{-1}\tilde{F}_1^{\cdot}, \qquad E_2^{\cdot} = t^{-1}\tilde{F}_2^{\cdot}$$

and

$$S_1 \cap S_2 \supset t^{-1}(\tilde{T}_1 \cap \tilde{T}_2)$$

it follows that

$$\mathrm{ch}_{S_1 \cap S_2}(E_1^{\cdot} \otimes E_2^{\cdot}) = t^* \, \mathrm{ch}_{\tilde{T}_1 \cap \tilde{T}_2}(\tilde{F}_1^{\cdot} \otimes \tilde{F}_2^{\cdot}).$$

By Lemma 4.5.5, we know that the canonical map

$$\tilde{i} : \widehat{\mathrm{H}}^{\cdot}_{\tilde{T}_1 \cap \tilde{T}_2}(\tilde{C}; \mathbb{Q}) \to \widehat{\mathrm{H}}^{\cdot}(\tilde{C}; \mathbb{Q})$$

is injective. Since

$$\begin{aligned}
\tilde{i}(\mathrm{ch}_{\tilde{T}_1 \cap \tilde{T}_2}(\tilde{F}_1^{\cdot} \otimes \tilde{F}_2^{\cdot})) &= \mathrm{ch}(\tilde{F}_1^{\cdot} \otimes \tilde{F}_2^{\cdot}) \\
&= \mathrm{ch}\, \tilde{F}_1^{\cdot} \smile \mathrm{ch}\, \tilde{F}_2^{\cdot} \\
&= \tilde{i}(\mathrm{ch}_{T_1}(\tilde{F}_1^{\cdot}) \smile \mathrm{ch}_{T_2}(\tilde{F}_2^{\cdot}))
\end{aligned}$$

we get that

$$\mathrm{ch}_{\tilde{T}_1 \cap \tilde{T}_2}(\tilde{F}_1^{\cdot} \otimes \tilde{F}_2^{\cdot}) = \mathrm{ch}_{T_1}(\tilde{F}_1^{\cdot}) \smile \mathrm{ch}_{T_2}(\tilde{F}_2^{\cdot})$$

and the conclusion follows since

$$\mathrm{ch}_{S_1}(E_1^{\cdot}) = t^* \, \mathrm{ch}_{T_1}(\tilde{F}_1^{\cdot})$$

and

$$\mathrm{ch}_{S_2}(E_2^{\cdot}) = t^* \, \mathrm{ch}_{T_2}(\tilde{F}_2^{\cdot}).$$

□

4.6 Extension to coherent analytic sheaves

Let X be a topological space and let $\mathcal{R}$ be a sheaf of ring on X. Recall that $\mathcal{R}$ is coherent if the kernel of any morphism of the type

$$\mathcal{R}^p_{|U} \to \mathcal{R}_{|U} \quad (U \text{ open subset of } X)$$

is locally of finite type. In this case, a $\mathcal{R}$-module $\mathcal{F}$ is coherent if and only if it admits locally presentations of the form

$$\mathcal{R}^{p_1} \to \mathcal{R}^{p_0} \to \mathcal{F} \to 0.$$

Moreover, these sheaves form an abelian category denoted $\mathcal{C}\mathrm{oh}(\mathcal{R})$. When X (resp. M) is a complex (resp. real) analytic manifold, $\mathcal{O}_X$ (resp. $\mathcal{A}_M$) is a coherent sheaf of rings and one sets for short

$$\mathcal{C}\mathrm{oh}(X) = \mathcal{C}\mathrm{oh}(\mathcal{O}_X) \quad (\text{resp. } \mathcal{C}\mathrm{oh}(M) = \mathcal{C}\mathrm{oh}(\mathcal{A}_M)\ )$$

and calls coherent analytic sheaves the objects of this category. Note that among coherent analytic sheaves on X (resp. M), one finds the locally free $\mathcal{O}_X$-modules (resp. $\mathcal{A}_X$-modules). They correspond to the sheaves of complex (resp. real) analytic sections of complex (resp. real) analytic complex vector bundles on X and form an additive subcategory of $\mathcal{C}\mathrm{oh}(X)$ (resp. $\mathcal{C}\mathrm{oh}(M)$) which will be denoted $\mathcal{B}\mathrm{nd}(X)$ (resp. $\mathcal{B}\mathrm{nd}(M)$).

Although we will not review in details the theory of coherent analytic sheaves here, we will feel free to use its main results without proof since the interested reader can find them in standard texts (see e.g. [12, 13]).

Definition 4.6.1. Let M be a real analytic manifold and let S be a closed subset of M. We denote $\mathcal{K}^{\mathrm{b}}_S(\mathcal{B}\mathrm{nd}(M))$ (resp. $\mathcal{D}^{\mathrm{b}}_S(\mathcal{C}\mathrm{oh}(M))$) the full triangulated subcategory of $\mathcal{K}^{\mathrm{b}}(\mathcal{B}\mathrm{nd}(M))$ (resp. $\mathcal{D}^{\mathrm{b}}(\mathcal{C}\mathrm{oh}(M))$) formed by the complexes which are exact on $M \setminus S$.

Lemma 4.6.2. *Assume M is a compact real analytic manifold. Then,*

(a) For any $\mathcal{F}^\cdot \in \mathcal{C}^{\mathrm{b}}(\mathcal{C}\mathrm{oh}(M))$ there is $\mathcal{L}^\cdot \in \mathcal{C}^{\mathrm{b}}(\mathcal{B}\mathrm{nd}(M))$ and a quasi-isomorphism

$$u^\cdot : \mathcal{E}^\cdot \to \mathcal{F}^\cdot.$$

(b) Any exact sequence

$$0 \to \mathcal{E}^p \to \ldots \to \mathcal{E}^q \to 0$$

with $\mathcal{E}^p$, …, $\mathcal{E}^q$ in $\mathcal{B}\mathrm{nd}(M)$ splits.

(c) *For any closed subset S of M, the canonical inclusion*

$$\mathcal{B}\mathrm{nd}(M) \to \mathcal{C}\mathrm{oh}(M)$$

induces the equivalence of triangulated categories

$$\mathcal{K}^{\mathrm{b}}_S(\mathcal{B}\mathrm{nd}(M)) \to \mathcal{D}^{\mathrm{b}}_S(\mathcal{C}\mathrm{oh}(M)).$$

Proof. (a) As is well known (see e.g. [18, Corollary 1.7.8]), it is sufficient to prove that:

(i) For any coherent analytic sheaf $\mathcal{F}$ on M, there is an epimorphism

$$\mathcal{E} \twoheadrightarrow \mathcal{F}$$

with $\mathcal{E} \in \mathcal{B}\mathrm{nd}(M)$.

(ii) There is an integer n such that for any exact sequence

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \ldots \to \mathcal{E}_0$$

of $\mathcal{C}\mathrm{oh}(M)$ with $\mathcal{E}_{n-1}, \ldots, \mathcal{E}_0$ in $\mathcal{B}\mathrm{nd}(M)$ we have $\mathcal{E}_n \in \mathcal{B}\mathrm{nd}(M)$.

Property (i) may be established as follows. Let $\mathcal{F}$ be a coherent analytic sheaf on M. Since M has a complex neighborhood which is a Stein manifold, Cartan's theorem A shows that $\mathcal{F}$ is generated by its global sections. Using the compactness of M and the fact that $\mathcal{F}$ is locally of finite type, one finds global sections $s_1, \ldots, s_N$ of $\mathcal{F}$ inducing an epimorphism

$$\mathcal{A}^N_M \twoheadrightarrow \mathcal{F}.$$

It follows that any object of $\mathcal{C}\mathrm{oh}(M)$ is a quotient of an object of $\mathcal{B}\mathrm{nd}(M)$.

Let us now show that property (ii) holds with n equal to the dimension of M. Let

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \ldots \to \mathcal{E}_0$$

be an exact sequence of $\mathcal{C}\mathrm{oh}(M)$ with $\mathcal{E}_{n-1}, \ldots, \mathcal{E}_0$ in $\mathcal{B}\mathrm{nd}(M)$. Denote $\mathcal{F}$ the cokernel of the last morphism. The global homological dimension of $\mathcal{A}_M$ being equal to n, the flat dimension of $\mathcal{F}$ is not greater than n. Since $\mathcal{E}_{n-1}, \ldots, \mathcal{E}_0$ are flat $\mathcal{A}_M$-modules, it follows that $\mathcal{E}_n$ is also flat. Applying Nakayama's lemma, we see that $\mathcal{E}_n$ is locally free and hence belongs to $\mathcal{B}\mathrm{nd}(M)$ as expected.

(b) Let

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0 \qquad (*)$$

be an exact sequence with $\mathcal{E}$ and $\mathcal{E}''$ in $\mathcal{B}\mathrm{nd}(M)$. Since $\mathcal{E}''$ is locally free, we have

$$R\mathcal{H}om_{\mathcal{A}_M}(\mathcal{E}'', \mathcal{E}') \simeq \mathcal{H}om_{\mathcal{A}_M}(\mathcal{E}'', \mathcal{E}').$$

Moreover, $\mathcal{H}om_{\mathcal{A}_M}(\mathcal{E}'', \mathcal{E}')$ being a coherent analytic sheaf, Cartan's theorem B shows that

$$\mathrm{R}\Gamma(M; \mathcal{H}om_{\mathcal{A}_M}(\mathcal{E}'', \mathcal{E}')) \simeq \Gamma(M; \mathcal{H}om_{\mathcal{A}_M}(\mathcal{E}'', \mathcal{E}')).$$

Therefore

$$\mathrm{RHom}_{\mathcal{A}_M}(\mathcal{E}'', \mathcal{E}') \simeq \mathrm{Hom}_{\mathcal{A}_M}(\mathcal{E}'', \mathcal{E}')$$

and $\mathcal{E}''$ is a projective $\mathcal{A}_M$-module. In particular, the sequence (*) splits. Since a direct summand of a flat module is flat, $\mathcal{E}'$ is flat and hence locally free. With these results at hand, we can conclude by a simple iteration procedure.

(c) This follows directly from (a) and (b) thanks to well-known results of homological algebra. □

Proposition 4.6.3. *Let M be a compact real analytic manifold and let S be a closed subset of M. There is a unique way to define a local Chern character*

$$\mathrm{ch}_S : \mathcal{D}^{\mathrm{b}}_S(\mathcal{C}\mathrm{oh}(M)) \to \mathrm{H}^{\cdot}_S(M; \mathbb{Z})$$

which is invariant by isomorphism and such that

$$\mathrm{ch}_S(\mathcal{E}^{\cdot}) = \mathrm{ch}_S(E^{\cdot})$$

if $\mathcal{E}^{\cdot}$ is an object of $\mathcal{K}^{\mathrm{b}}_S(\mathcal{B}\mathrm{nd}(M))$ and $E^{\cdot}$ is the associated complex of complex vector bundles. Moreover, such a local chern character is additive and multiplicative.

Proof. Thanks to Proposition 4.5.2 and Proposition 4.5.7, the result follows directly from the preceding lemma. □

Remark 4.6.4. It follows easily from the additivity of the local Chern character introduced in the preceding proposition that

$$\mathrm{ch}_S(\mathcal{F}^{\cdot}) = \sum_{k \in \mathbb{Z}} (-1)^k \,\mathrm{ch}_S(\mathrm{H}^k(\mathcal{F})).$$

So, the local Chern character for complexes of coherent analytic sheaves may be reduced to the local Chern character for coherent analytic sheaves. A similar reduction has however no meaning for complexes of complex vector bundles since the corresponding cohomology sheaves are not locally free in general.

Definition 4.6.5. Let M be a compact real analytic manifold, let S be a closed subset of M and let $\mathcal{F}^{\cdot}$ be an object of $\mathcal{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_M)$ (i.e. a complex of $\mathcal{A}_M$-modules with bounded coherent cohomology). Assume $\mathcal{F}^{\cdot}$ is supported by S. We extend the definition of ch_S by setting

$$\mathrm{ch}_S(\mathcal{F}^{\cdot}) = \sum_{k\in\mathbb{Z}} (-1)^k \mathrm{ch}_S(\mathrm{H}^k(\mathcal{F}))$$

where the classes

$$\mathrm{ch}_S(\mathrm{H}^k(\mathcal{F})) \in \mathrm{H}^{\cdot}_S(M;\mathbb{Z})$$

are defined according to the preceding proposition.

Proposition 4.6.6. *Let M be a compact real analytic manifold and let S and T be closed subsets of M.*

(a) *Assume*

$$\mathcal{E}^{\cdot} \to \mathcal{F}^{\cdot} \to \mathcal{G}^{\cdot} \xrightarrow{+1}$$

is a distinguished triangle of $\mathcal{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_M)$ which is supported by S, then

$$\mathrm{ch}_S(\mathcal{F}^{\cdot}) = \mathrm{ch}_S(\mathcal{E}^{\cdot}) + \mathrm{ch}_S(\mathcal{G}^{\cdot}).$$

(b) *Assume $\mathcal{E}^{\cdot}$ and $\mathcal{F}^{\cdot}$ are objects of $\mathcal{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_M)$ supported respectively by S and T. Then $\mathcal{E}^{\cdot} \otimes^L_{\mathcal{A}_M} \mathcal{F}^{\cdot}$ is an object of $\mathcal{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_M)$ supported by $S \cap T$ and*

$$\mathrm{ch}_{S\cap T}(\mathcal{E}^{\cdot} \otimes^L_{\mathcal{A}_M} \mathcal{F}^{\cdot}) = \mathrm{ch}_S(\mathcal{E}^{\cdot}) \smile \mathrm{ch}_T(\mathcal{F}^{\cdot})$$

Proof. Part (a) follows from the long exact sequence of cohomology and the additivity of the local Chern character for coherent analytic sheaves. Part (b) follows from the multiplicativity of the local Chern character of Proposition 4.6.3 combined with the fact that any object of $\mathcal{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_M)$ is isomorphic in $\mathcal{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A}_M)$ with an object of $\mathcal{D}^{\mathrm{b}}(\mathcal{C}\mathrm{oh}(M))$. $\square$

Definition 4.6.7. Let X be a compact complex manifold and let S be a closed subset of X. Assume $\mathcal{F}^{\cdot}$ is an object of $\mathcal{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X)$ supported by S. Using the real analytic structure of X, we set

$$\mathrm{ch}_S(\mathcal{F}) = \mathrm{ch}_S(\mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{F}).$$

Remark 4.6.8. Since $\mathcal{A}_X$ is flat over $\mathcal{O}_X$, the local Chern character introduced in the preceding proposition is clearly invariant by isomorphism, additive and multiplicative.

5

Riemann-Roch theorem

5.1 Introduction

Let X be a compact complex analytic manifold of complex dimension n. To fix the notations, let us recall that

$\mathcal{O}_X$	$=$	sheaf of holomorphic functions on X
Ω^p_X	$=$	sheaf of holomorphic p-forms on X
$\mathcal{K}_X$	$=$	sheaf of meromorphic functions on X
$\mathcal{A}^{(p,q)}_X$	$=$	sheaf of real analytic forms of bitype (p,q)
$\mathcal{B}^{(p,q)}_X$	$=$	sheaf of hyperfunction forms of bitype (p,q)
$\mathcal{C}^{(p,q)}_{\infty,X}$	$=$	sheaf of smooth forms of bitype (p,q)
$\mathcal{D}b^{(p,q)}_X$	$=$	sheaf of distribution forms of bitype (p,q)
$\mathcal{A}^n_X$	$=$	sheaf of real analytic n-forms
$\mathcal{B}^n_X$	$=$	sheaf of hyperfunction n-forms
$\mathcal{C}^n_{\infty,X}$	$=$	sheaf of smooth n-forms
$\mathcal{D}b^n_X$	$=$	sheaf of distribution n-forms

These various sheaves enter in the following well-known resolutions.

Proposition 5.1.1.

Real de Rham resolutions. *The sequence*

$$0 \to \mathbb{C}_X \to \mathcal{C}^0_{\infty,X} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{C}^{2n}_{\infty,X} \to 0$$

is exact and we get analogous exact sequences if we replace $\mathcal{C}_{\infty,X}$ by $\mathcal{A}_X$, $\mathcal{B}_X$ or $\mathcal{D}b_X$.

Dolbeault resolutions. *The sequence*

$$0 \to \Omega^p_X \to \mathcal{C}^{(p,0)}_{\infty,X} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{C}^{(p,n)}_{\infty,X} \to 0$$

is exact and we get analogous exact sequences if we replace $\mathcal{C}_{\infty,X}$ by $\mathcal{A}_X$, $\mathcal{B}_X$ or $\mathcal{D}b_X$.

Complex de Rham resolution. *The sequence*

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_X \to 0$$

is exact.

Since X is a canonically oriented compact topological manifold of dimension $2n$, we have the two following propositions.

Proposition 5.1.2 (Topological finiteness). *We have*

$$\mathrm{H}^k(X; \mathbb{C}_X) = 0$$

if $k \notin [0, 2n]$ and

$$\dim \mathrm{H}^k(X; \mathbb{C}_X) < +\infty$$

for any $k \in [0, 2n]$.

Remark 5.1.3. It follows from the preceding proposition that the complex Betti numbers

$$b^k(X) = \dim \mathrm{H}^k(X; \mathbb{C}_X) \quad (k \in \mathbb{Z})$$

are well-defined topological invariants associated to X. Another topological invariant of X is its Euler-Poincaré characteristic. Note that it follows from the universal coefficient formula that

$$\chi(X) = \sum_{k \in \mathbb{Z}} (-1)^k b^k(X).$$

Proposition 5.1.4 (Topological duality). *By Poincaré duality, we have an isomorphism*
$$\mathrm{H}^k(X;\mathbb{C}_X)^* \simeq \mathrm{H}^{2n-k}(X;\mathbb{C}_X)$$
which can be made more explicit by means of the pairing
$$\langle\cdot,\cdot\rangle : \Gamma(X;\mathcal{D}b_X^{2n-k}) \otimes \Gamma_c(X;\mathcal{C}_{\infty,X}^k) \to \mathbb{C}$$
defined by the formula
$$\langle u,\omega\rangle = \int u \wedge \omega.$$

Remark 5.1.5. It follows directly from the preceding proposition that
$$b^{2n-k}(X) = b^k(X)$$
for any $k \in \mathbb{Z}$. In particular,
$$\chi(X) = (-1)^n b^n(X) + 2\sum_{k=0}^{n-1}(-1)^k b^k(X)$$
so that
$$\chi(X) \equiv b^n(X) \pmod 2.$$

Finiteness and duality results also holds for coherent analytic sheaves.

Proposition 5.1.6. *For any coherent analytic sheaf $\mathcal{F}$ on X, we have*
$$\mathrm{H}^k(X;\mathcal{F}) = 0$$
if $k \notin [0,n]$ and
$$\dim \mathrm{H}^k(X;\mathcal{F}) < +\infty$$
if $k \in [0,n]$.

Proof. We will only give a sketch of the proof for the case where $\mathcal{F}$ is locally free. For the general case, we refer to standard texts on complex analytic geometry.

By tensorization on $\mathcal{O}_X$ with the Dolbeault resolution
$$0 \to \mathcal{O}_X \to \mathcal{C}_{\infty,X}^{(0,0)} \to \cdots \to \mathcal{C}_{\infty,X}^{(0,n)} \to 0$$
we obtain the resolution
$$0 \to \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{C}_{\infty,X}^{(0,0)} \to \cdots \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{C}_{\infty,X}^{(0,n)} \to 0.$$

Since $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{C}^{(0,k)}_{\infty,X}$ is locally isomorphic to a direct sum of a finite number of copies of $\mathcal{C}^{(0,k)}_{\infty,X}$, it is a soft sheaf. Therefore,

$$\mathrm{R}\Gamma(X;\mathcal{F}) \simeq \Gamma(X;\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{C}^{(0,\cdot)}_{\infty,X}).$$

The vanishing part of the result follows easily. To obtain the finiteness part, we consider the resolution

$$0 \to \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}^{(0,0)}_X \to \cdots \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}^{(0,n)}_X \to 0$$

obtained by applying the exact functor $\mathcal{F} \otimes_{\mathcal{O}_X} \cdot$ to the Dolbeault resolution

$$0 \to \mathcal{O}_X \to \mathcal{A}^{(0,0)}_X \to \cdots \to \mathcal{A}^{(0,n)}_X \to 0.$$

Since $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}^{(0,k)}_X$ is a coherent real analytic sheaf on X, it is acyclic. Hence,

$$\mathrm{R}\Gamma(X;\mathcal{F}) \simeq \Gamma(X;\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}^{(0,\cdot)}_X)$$

and the canonical morphism

$$\Gamma(X;\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{A}^{(0,\cdot)}_X) \to \Gamma(X;\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{C}^{(0,\cdot)}_{\infty,X})$$

is a quasi-isomorphism. One checks easily that the components of the first complex are (DFS) spaces and that the components of the second complex are (F) spaces. Since the morphism from the left complex to the right complex is clearly continuous, the conclusion follows from the next lemma. □

Lemma 5.1.7. *Let*

$$u^{\cdot} : E^{\cdot} \to F^{\cdot}$$

be a morphism of complexes of locally convex topological vector spaces. Assume that for any $k \in \mathbb{Z}$*,*

(i) E^k *is a (DFS) space;*

(ii) F^k *is a (F) space;*

(iii) $\mathrm{H}^k(u^{\cdot})$ *is surjective.*

Then, $\mathrm{H}^k(F^{\cdot})$ *is finite dimensional.*

Sketch of proof. The basic idea is to write $E^k = \varinjlim_{i\in\mathbb{N}} E^k_i$ where E^k_i is a Banach space; the transitions being compact and then to use Baire's theorem to reduce the result to Schwartz' compact perturbation Lemma. □

Proposition 5.1.8. *For any coherent analytic sheaf $\mathcal{F}$ on X, we have the canonical isomorphisms*

$$\mathrm{H}^k(X;\mathcal{F})^* \simeq \mathrm{H}^{n-k}(X; R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\Omega_X))$$

where Ω_X denotes as usual the sheaf of holomorphic n-forms.

Proof. We will only sketch the proof in the case where $\mathcal{F}$ is locally free.

Working as in the proof of Proposition 5.1.6, we get the two isomorphisms

$$\mathrm{R}\Gamma(X;\mathcal{F}) \simeq \Gamma(X;\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{C}^{(0,\cdot)}_{\infty,X})$$

and

$$\mathrm{R}\Gamma(X; R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\Omega_X)) \simeq \Gamma(X; R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{D}b^{(n,\cdot)}_X)).$$

Consider the canonical pairing

$$[\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{C}^{(0,k)}_{\infty,X}]\otimes_{\mathbb{C}}[R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{D}b^{(n,n-k)}_X)] \to \mathcal{D}b^{(n,n)}_X$$

which sends $(f\otimes w, h)$ to $w\wedge h(f)$. Combining it with the integration of distributions, we get the pairing

$$\Gamma(X;\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{C}^{(0,k)}_{\infty,X})\otimes_{\mathbb{C}}\Gamma(X; R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{D}b^{(n,n-k)}_X)) \to \mathbb{C}$$

and hence the morphism of complexes

$$\Gamma(X;\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{C}^{(0,\cdot)}_{\infty,X})' \simeq \Gamma(X; R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{D}b^{(n,n-\cdot)}_X)).$$

Using the definition of distributions, one checks easily that this morphism is in fact an isomorphism. The cohomology of the complexes involved being finite dimensional, the differentials are strict and we get

$$\mathrm{H}^k(\Gamma(X;\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{C}^{(0,\cdot)}_{\infty,X}))^* \simeq \mathrm{H}^{n-k}(\Gamma(X; R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{D}b^{(n,\cdot)}_X))).$$

The conclusion follows. $\square$

Example 5.1.9. Since Ω^p_X is a coherent analytic sheaf, Proposition 5.1.6 shows that the Hodge numbers

$$h^{p,q}(X) = \dim \mathrm{H}^q(X;\Omega^p) \quad (q\in\mathbb{Z})$$

are well-defined holomorphic invariants of X which vanish for $q\notin[0,n]$. Of course, we have

$$\chi^p(X) := \chi(X;\Omega^p_X) = \sum_{q\in\mathbb{Z}}(-1)^q h^{p,q}.$$

Moreover, it follows from the complex de Rham resolution that

$$\chi(X) = \sum_{p\in\mathbb{Z}} (-1)^p \chi^p(X).$$

Note that the canonical morphism

$$\wedge : \Omega_X^p \otimes \Omega_X^{n-p} \to \Omega_X^n$$

induces an isomorphism

$$\Omega_X^{n-p} \simeq \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^p, \Omega_X^n).$$

Hence, applying the duality result for coherent analytic sheaves, we see that

$$h^{n-p,q} = h^{p,n-q}.$$

The $h^{p,q}$ have many other interesting properties. However, since they are related to Hodge theory we will not review them here.

Thanks to the results in this section, one sees that the Euler-Poincaré characteristic

$$\chi(X;\mathcal{F}) = \sum_{k=0}^{\infty} (-1)^k \dim \mathrm{H}^k(X;\mathcal{F})$$

makes sense for any coherent analytic sheaf on a compact complex analytic manifold. The computation of this number forms what one may call the generalized Riemann-Roch problem (its link with the original results of Riemann and Roch will be explained in Sections 2–4 below). In [15], Hirzebruch solved this problem when X is a projective manifold and $\mathcal{F}$ is the sheaf of complex analytic sections of a complex analytic vector bundle F of rank r by expressing the number $\chi(X;\mathcal{F})$ by means of a formula of the type

$$\chi(X;\mathcal{F}) = \int_X P(c_1(F), \cdots, c_r(F), c_1(TX), \cdots, c_n(TX))$$

where P is a polynomial with rational coefficients which depends only on r and n. In Sections 5–8 we will establish the following slightly more general result.

Theorem 5.1.10. *Let X be a projective complex analytic manifold of dimension n and let $\mathcal{F}$ be a coherent analytic sheaf on X. Then,*

$$\chi(X;\mathcal{F}) = \int_X \mathrm{ch}(\mathcal{F}) \smile \mathrm{td}(TX).$$

To better understand the link between this formula and the preceding one let us write it explicitly in terms of Chern classes in a two simple cases.

Case of line bundles on curves.

We need to compute $\mathrm{ch}(E)$ and $\mathrm{td}(TX)$ in degrees 0 and 2. We have

$$\frac{x}{1-e^{-x}} = \frac{x}{1-(1+(-x)+\frac{(-x)^2}{2}+o(x^2))} = \frac{1}{1-\frac{x}{2}+o(x)} = 1+\frac{x}{2}+o(x)$$

and

$$e^x = 1+x+o(x).$$

Therefore, using the fact that E and TX are line bundles, we get

$$\mathrm{ch}(E) = 1 + c_1(E)$$

$$\mathrm{td}(TX) = 1+\frac{c_1(TX)}{2}.$$

Hence,

$$[\mathrm{ch}(E)\smile \mathrm{td}(TX)]^2 = c_1(E)+\frac{c_1(TX)}{2}$$

so that

$$\chi(X;\mathcal{E}) = \int_X c_1(E)+\frac{c_1(TX)}{2}.$$

Case of line bundles on surfaces.

We need to compute $\mathrm{ch}(E)$ and $\mathrm{td}(TX)$ in degrees 0, 2 and 4. We have

$$\begin{aligned}\frac{x}{1-e^{-x}} &= \frac{x}{1-(1+(-x)+\frac{(-x)^2}{2}+\frac{(-x)^3}{6}+o(x^3))}\\ &= \frac{1}{1-\frac{x}{2}+\frac{x^2}{6}+o(x^2)}\\ &= 1+\frac{x}{2}+\frac{x^2}{12}+o(x^2)\end{aligned}$$

and

$$e^x = 1+x+\frac{x^2}{2}+o(x^2).$$

It follows that we have at order 2

$$
\begin{aligned}
\frac{x_1}{1-e^{-x_1}} \cdot \frac{x_2}{1-e^{-x_2}} &= \left(1+\frac{x_1}{2}+\frac{x_1^2}{12}\right)\left(1+\frac{x_2}{2}+\frac{x_2^2}{12}\right) \\
&= 1+\frac{x_2}{2}+\frac{x_2^2}{12}+\frac{x_1}{2}+\frac{x_1x_2}{4}+\frac{x_1^2}{12} \\
&= 1+\frac{x_1+x_2}{2}+\frac{x_1^2+3x_1x_2+x_2^2}{12} \\
&= 1+\frac{(x_1+x_2)}{2}+\frac{(x_1+x_2)^2+x_1x_2}{12} \\
&= 1+\frac{S_{2,1}(x_1,x_2)}{2}+\frac{S_{2,1}^2(x_1,x_2)+S_{2,2}(x_1,x_2)}{12}.
\end{aligned}
$$

Therefore,

$$
\operatorname{td}(TX) = 1+\frac{c_1(TX)}{2}+\frac{c_1(TX)^2+c_2(TX)}{12}
$$

and

$$
\operatorname{ch}(E) = 1+c_1(E)+\frac{c_1(E)^2}{2}.
$$

Finally,

$$
[\operatorname{ch}(E)\smile \operatorname{td}(TX)]^4 = \frac{c_1(TX)^2+c_2(TX)}{12}+\frac{c_1(E)c_1(TX)}{2}+\frac{c_1(E)^2}{2}
$$

and

$$
\chi(X;\mathcal{E}) = \int_X \frac{c_1(TX)^2+c_2(TX)}{12}+\frac{c_1(E)c_1(TX)}{2}+\frac{c_1(E)^2}{2}.
$$

5.2 Cohomology of compact complex curves

Let X be a connected compact (smooth) complex curve.

Proposition 5.2.1. *There is a natural integer g such that*

<table>
<tr><td>$\mathrm{H}^0(X;\mathcal{O}_X) \simeq \mathbb{C}$
$\mathrm{H}^1(X;\mathcal{O}_X) \simeq \mathbb{C}^g$
$\mathrm{H}^k(X;\mathcal{O}_X) \simeq 0 \quad (k\geq 2)$</td><td>$\mathrm{H}^0(X;\Omega_X) \simeq \mathbb{C}^g$
$\mathrm{H}^1(X;\Omega_X) \simeq \mathbb{C}$
$\mathrm{H}^k(X;\Omega_X) \simeq 0 \quad (k\geq 2)$</td></tr>
<tr><td colspan="2">$\mathrm{H}^0(X;\mathbb{C}_X) \simeq \mathbb{C}$
$\mathrm{H}^1(X;\mathbb{C}_X) \simeq \mathbb{C}^{2g}$
$\mathrm{H}^2(X;\mathbb{C}_X) \simeq \mathbb{C}$
$\mathrm{H}^k(X;\mathbb{C}_X) \simeq 0 \quad (k\geq 3)$</td></tr>
</table>

Proof. Writing the long exact sequence of cohomology associated to the short exact sequence

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \to \Omega_X \to 0,$$

we get the exact sequence

$$\begin{aligned}
0 \to \mathrm{H}^0(X;\mathbb{C}_X) &\to \mathrm{H}^0(X;\mathcal{O}_X) \to \mathrm{H}^0(X;\Omega_X) \to \\
\to \mathrm{H}^1(X;\mathbb{C}_X) &\to \mathrm{H}^1(X;\mathcal{O}_X) \to \mathrm{H}^1(X;\Omega_X) \to \\
\to \mathrm{H}^2(X;\mathbb{C}_X) &\longrightarrow 0.
\end{aligned}$$

Since X is connected, $\mathrm{H}^0(X;\mathbb{C}_X) \simeq \mathbb{C}$ and since X is compact, the maximum principle shows that $\mathrm{H}^0(X;\mathcal{O}_X) \simeq \mathbb{C}$. Therefore, the morphism

$$\mathrm{H}^0(X;\mathbb{C}_X) \to \mathrm{H}^0(X;\mathcal{O}_X) \tag{*}$$

is an isomorphism. By duality, we see that

$$\mathrm{H}^2(X;\mathbb{C}_X) \simeq \mathbb{C}, \qquad \mathrm{H}^1(X;\Omega_X) \simeq \mathbb{C}$$

and that the morphism

$$\mathrm{H}^1(X;\Omega_X) \to \mathrm{H}^2(X;\mathbb{C}_X)$$

is the dual of the isomorphism (*). It follows that the sequence

$$0 \to \mathrm{H}^0(X;\Omega_X) \to \mathrm{H}^1(X;\mathbb{C}_X) \to \mathrm{H}^1(X;\mathcal{O}_X) \to 0$$

is exact. Since

$$\mathrm{H}^1(X;\mathcal{O}_X) \simeq \mathrm{H}^0(X;\Omega_X)^*,$$

the spaces $\mathrm{H}^1(X;\mathcal{O}_X)$ and $\mathrm{H}^0(X;\Omega_X)$ have the same dimension g. It follows that $\mathrm{H}^1(X;\mathcal{O}_X) \simeq \mathbb{C}^g$, $\mathrm{H}^0(X;\Omega_X) \simeq \mathbb{C}^g$ and that $\mathrm{H}^1(X;\mathbb{C}_X) \simeq \mathbb{C}^{2g}$. □

Definition 5.2.2. The integer g which appears in the preceding table is called the *genus* of the Riemann surface X. One defines it classically as the maximal number of linearly independent holomorphic 1-forms.

Remark 5.2.3. Note that since $2g$ is the first Betti number of X, g is a topological invariant of X. One can show that this is the only invariant of this kind. Note also that it follows from the cohomology table of X that

$$\chi(X) = 2 - 2g.$$

5.3 Divisors on complex curves

Let A be a commutative integral ring and let K be its field of quotients. Denote $A^\times$ the multiplicative monoid formed by the non-zero elements of A. Define the divisibility relation "$|$" on $A^\times$ by

$$a|b \iff \exists c \in A^\times \quad \text{with} \quad b = a \cdot c.$$

This is clearly a preorder relation compatible with the multiplication of $A^\times$. Since A is integral, any element of $A^\times$ is cancelable and $A^\times$ has an associated group. This group may be identified with the group K^* of invertible (i.e. non-zero) elements of K. The relation "$|$" induces a preorder relation on K^* defined by

$$f|g \iff \exists c \in A^\times \quad \text{with} \quad g = f \cdot c \quad \iff \quad g/f \in A^\times.$$

Remark that if f, $g \in K^*$ are such that

$$f|g \qquad \text{and} \qquad g|f$$

then $g/f \in A^*$ and conversely. If follows that $D = K^*/A^*$ has a natural structure of ordered commutative group. We denote $+$ the group operation of D and we denote $\leq$ its canonical order relation. We denote (f) the element of D associated to $f \in K^*$. The ordered group $(D, +, \leq)$ is called the *group of principal divisors of the ring* A. This group enters in the exact sequence

$$1 \to A^* \to K^* \to D \to 0.$$

In the case of a complex curve X, we may consider by analogy the sheaves of commutative groups $\mathcal{O}_X^*$ (resp. $\mathcal{K}_X^*$) of invertible elements of $\mathcal{O}_X$ (resp. $\mathcal{K}_X$) and we may define the sheaf of abelian groups $\mathcal{D}\mathrm{iv}_X$ as the cokernel of the canonical inclusion

$$\mathcal{O}_X^* \to \mathcal{K}_X^*.$$

It follows from the exact sequence

$$1 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{D}\mathrm{iv}_X \to 0$$

that $(\mathcal{D}\mathrm{iv}_X)_x$ is the group of principal divisors of the integral ring $(\mathcal{O}_X)_x$. So, it is canonically endowed with an order relation $(\leq)_x$. Hence, we get an order relation $\leq$ on the sections of $\mathcal{D}\mathrm{iv}_X$. A *divisor on* X is the data of a global section D of $\mathcal{D}\mathrm{iv}_X$. A divisor coming from a global section f of $\mathcal{K}_X^*$ is called *principal*, we denote it (f). It follows from the construction of $\mathcal{D}\mathrm{iv}_X$

that a divisor on X is locally principal. It is however not globally principal in general. To study this phenomenon, one introduces the quotient group

$$\Gamma(X; \mathcal{D}\mathrm{iv}_X)/\Gamma(X; \mathcal{K}_X^*).$$

This group is called the *Picard group of* X and denoted $\mathrm{Pic}(X)$. Two divisor which have the same image in $\mathrm{Pic}(X)$ are said to be equivalent.

The structure of $(\mathcal{D}\mathrm{iv}_X)_x$ is very simple. As a matter of fact, if t is a local coordinate such that $t(x) = 0$, any meromorphic function f may be uniquely written in a neighborhood of x as

$$f = t^n h$$

where $n \in \mathbb{Z}$ and $h \in \mathcal{O}_{X,x}^*$ (consequence of Taylor formula). It follows that $(f) = n(t)$. Moreover, we see easily that if t' is another local coordinate such that $t'(x) = 0$, we have $(t) = (t')$. Therefore, (t) is a canonically defined divisor supported by $\{x\}$. We denote it by $[x]$ and we denote $\mathrm{ord}_x(f)$ the unique integer n such that

$$(f) = n[x].$$

We may sum up what precedes by saying that

$$\mathrm{ord}_x : (\mathcal{D}\mathrm{iv}_X)_x \to \mathbb{Z}$$

is an isomorphism of ordered abelian groups. Any divisor D of X being locally principal, $\{x : \mathrm{ord}_x(D) \neq 0\}$ is locally finite; hence finite (X is compact). It follows that

$$D = \sum_{x \in \mathrm{supp}\, D} \mathrm{ord}_x(D)[x].$$

Thus, there is a bijection between the ordered group of divisors of X and the ordered group

$$\mathbb{Z}^{(X)}.$$

We define the *degree* of the divisor D by the formula

$$\deg D = \sum_{x \in \mathrm{supp}\, D} \mathrm{ord}_x(D).$$

To a divisor D on X, one associates the sheaf $\mathcal{O}_X(D)$ defined by setting

$$\mathcal{O}_X(D)(U) = \{f \in \mathcal{K}_X^*(U) : (f) \geq -D|_U\} \cup \{0\}$$

for any connected open subset U of X. One checks easily that $\mathcal{O}_X(D)$ is an $\mathcal{O}_X$-submodule of $\mathcal{K}_X$. Consider a point $x \in X$. This point has a connected neighborhood U on which D is principal. So, we find a meromorphic function g on U such that $D|_U = (g)$. It follows that $f \in \mathcal{O}_X(D)(U)$ if and only if $f = 0$ or $(f) \geq -(g)$. Hence,

$$\mathcal{O}_X(D)(U) = \{\frac{h}{g} : h \in \mathcal{O}_X(U)\}$$

and $\mathcal{O}_X(D)|_U$ is the free $\mathcal{O}_X$-submodule of rank 1 of $\mathcal{K}_X$ generated by $1/g$. Therefore, $\mathcal{O}_X(D)$ is a locally free $\mathcal{O}_X$-module of rank 1. Remark that if D, D' are two divisors on X such that

$$D' = D + (g)$$

with $g \in \Gamma(X; \mathcal{K}_X^*)$, then

$$\begin{aligned} \mathcal{O}_X(D') &\to \mathcal{O}_X(D) \\ f &\mapsto fg \end{aligned}$$

is an isomorphism of $\mathcal{O}_X$-modules. Therefore, the isomorphy class of $\mathcal{O}_X(D)$ depends only on the class of D in $\mathrm{Pic}(X)$.

5.4 Classical Riemann and Roch theorems

Definition 5.4.1. Let $\mathcal{L}$ be a locally free $\mathcal{O}_X$-module of rank 1 on X. Consider a non-zero meromorphic section s of $\mathcal{L}$, i.e. assume that

$$s \in \Gamma(X; \mathcal{K}_X \otimes_{\mathcal{O}_X} \mathcal{L}) \setminus \{0\}.$$

Consider a neighborhood U of $x \in X$ on which $\mathcal{L}$ is isomorphic to $\mathcal{O}_X$ and denote l a generator of $\mathcal{L}|_U$. Clearly, we have

$$s|_U = f \cdot l$$

where f is a meromorphic function on U uniquely determined by this relation. If l' is another generator of $\mathcal{L}|_U$, we have $l = hl'$ where $h \in \mathcal{O}_X^*(U)$ and

$$s|_U = f' \cdot l'$$

where $f' \in \mathcal{K}_X(U)$ is given by $f' = fh$. It follows that $(f) = (f')$ and that this divisor depends only on $s|_U$. We denote it $(s|_U)$. Using a gluing process, we obtain a well-defined divisor (s) on X.

From this definition it follows easily that

Proposition 5.4.2. *We have an $\mathcal{O}_X$-linear isomorphism*

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D) \simeq \mathcal{L}(D)$$

where $\mathcal{L}(D)$ is the subsheaf of $\mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{L}$ defined by setting

$$\mathcal{L}(D)(U) = \{s \in \Gamma(U; \mathcal{K}_X \otimes_{\mathcal{O}_X} \mathcal{L}) \setminus \{0\} : (s) \geq -D\} \cup \{0\}$$

for any connected open subset U of X.

Proposition 5.4.3. *For any locally free $\mathcal{O}_X$-module $\mathcal{L}$ of rank 1 and any divisor D on X, we have*

$$\chi(X; \mathcal{L}(D)) = \chi(X; \mathcal{L}) + \deg D.$$

In particular,

$$\dim \mathrm{H}^0(X; \mathcal{L}(D)) \geq \chi(X; \mathcal{L}) + \deg D.$$

Proof. Let D and D' be two divisors on X such that $D \leq D'$. By definition, it is clear that $\mathcal{O}_X(D)$ is a subsheaf of $\mathcal{O}_X(D')$. Denote $\mathcal{Q}$ the quotient sheaf $\mathcal{O}_X(D')/\mathcal{O}_X(D)$. Write D, D' as

$$D = \sum_{j=1}^{m} n_j [x_j], \qquad D' = \sum_{j=1}^{m} n'_j [x_j]$$

where $x_1, \cdots, x_m$ are points of X and $n_1, \ldots, n_m; n'_1, \ldots, n'_m$ are elements of $\mathbb{Z}$ (which may be equal to 0). One sees easily that $\mathcal{Q}_x = 0$ if $x \notin \{x_1, \ldots, x_m\}$. Now, consider a coordinate neighborhood U of a point $x_j \in \{x_1, \cdots, x_m\}$ such that $U \cap \{x_1, \cdots, x_m\} = \{x_j\}$. Let t be a coordinate on U such that $t(x_j) = 0$. Since $D|_U = n_j(t)$ and $D'|_U = n'_j(t)$, we see that

$$\mathcal{O}_X(D)_{x_j} = \{\frac{h}{t^{n_j}} : h \in (\mathcal{O}_X)_{x_j}\}, \qquad \mathcal{O}_X(D')_{x_j} = \{\frac{h'}{t^{n'_j}} : h' \in (\mathcal{O}_X)_{x_j}\}.$$

Since $n'_j \geq n_j$, we may write

$$h' = a_0 + a_1 t + \cdots + a_{n'_j - n_j - 1} t^{n'_j - n_j - 1} + t^{n'_j - n_j} h''$$

where $a_0, \cdots, a_{n'_j - n_j - 1} \in \mathbb{C}$ and $h'' \in (\mathcal{O}_X)_j$ are uniquely determined by this condition. It follows that

$$\mathcal{Q}_{x_j} = (\mathcal{O}_X(D')/\mathcal{O}_X(D))_{x_j} \simeq \mathbb{C}^{n'_j - n_j}.$$

The sheaf $\mathcal{Q}$ is thus supported by $\{x_1, \cdots, x_m\}$ and we have

$$\dim \mathcal{Q}_{x_j} = n'_j - n_j \qquad (1 \leq j \leq m).$$

Applying $\mathcal{L} \otimes_{\mathcal{O}_X} \cdot$ to the exact sequence

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D') \to \mathcal{Q} \to 0$$

and taking Euler-Poincaré characteristics, we see that

$$\chi(X; \mathcal{L}(D)) - \chi(X; \mathcal{L}(D')) + \chi(X; \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{Q}) = 0$$

and hence that

$$\chi(X; \mathcal{L}(D)) - \chi(X; \mathcal{L}(D')) + \deg D' - \deg D = 0.$$

Therefore,

$$\chi(X; \mathcal{L}(D')) - \deg D' = \chi(X; \mathcal{L}(D)) - \deg D.$$

If D, D' are divisors on X, there is a divisor $D'' \geq D$, $D'' \geq D'$. It follows from what precedes that

$$\chi(X; \mathcal{L}(D)) - \deg D$$

does not depend on D. The conclusion follows since

$$\mathcal{L}(0) \simeq \mathcal{L}.$$

□

Corollary 5.4.4. *For any divisor D on X, we have*

$$\chi(X; \mathcal{O}_X(D)) = \deg D + 1 - g.$$

In particular, $\deg D$ *depends only on the class of D in* $\mathrm{Pic}(X)$ *and*

$$\dim \mathrm{H}^0(X; \mathcal{O}_X(D)) \geq \deg D + 1 - g.$$

Proof. This follows directly from the preceding proposition since

$$\chi(X; \mathcal{O}_X) = \dim \mathrm{H}^0(X; \mathcal{O}_X) - \dim \mathrm{H}^1(X; \mathcal{O}_X) = 1 - g.$$

□

Remark 5.4.5. Let $x_1, \cdots, x_m$ be distinct points of X. The problem which led to the classical Riemann and Roch theorems was the computation of the dimension of the space of meromorphic functions which are holomorphic on $X \setminus \{x_1, \cdots, x_m\}$ and have poles of order lower or equal to 1 at $x_1, \cdots, x_m$. One checks easily that this dimension is nothing but

$$\dim \mathrm{H}^0(X; \mathcal{O}_X(D))$$

for $D = [x_1] + \cdots + [x_m]$. Therefore, the preceding corollary contains Riemann's inequality in its original form.

Proposition 5.4.6. *Let $\mathcal{L}$ be a locally free $\mathcal{O}_X$-module of rank 1. Fix $x \in X$. Then, $\mathcal{L}$ has meromorphic sections whose only pole is at x. In particular, $\mathcal{L}$ has non-zero meromorphic sections.*

Proof. In the preceding proposition, set $D = m[x]$ and choose m large enough in order that $\chi(X;\mathcal{L}) + \deg D > 0$. □

Corollary 5.4.7. *Let $\mathcal{L}$ be a locally free $\mathcal{O}_X$-module of rank 1. Then, there is a divisor D on X and an isomorphism*

$$\mathcal{O}_X(D) \simeq \mathcal{L}.$$

Moreover, although D is not unique its class in $\mathrm{Pic}(X)$ *depends only on $\mathcal{L}$.*

Proof. Let s be a non-zero meromorphic section of $\mathcal{L}$. Set $D = (s)$. Working on the open subsets where $\mathcal{L}$ is trivial, we see easily that

$$\begin{aligned} \mathcal{O}_X(D) &\to \mathcal{L} \\ f &\mapsto fs \end{aligned}$$

is a well-defined isomorphism. To see that the class of D in $\mathrm{Pic}(X)$ depends only on $\mathcal{L}$, it is sufficient to note that non-zero meromorphic sections s, s' of $\mathcal{L}$ are always linked by a relation of the type

$$s' = gs$$

with $g \in \Gamma(X;\mathcal{K}_X^*)$ and that such a relation entails that

$$(s') = (s) + (g).$$

□

Definition 5.4.8. Let $\mathcal{L}$ be a locally free $\mathcal{O}_X$ module of rank 1 on X. We denote $[\mathcal{L}]$ the class in $\mathrm{Pic}(X)$ of any divisor D such that

$$\mathcal{L} \simeq \mathcal{O}_X(D).$$

Example 5.4.9. Since Ω_X is a locally free $\mathcal{O}_X$-module of rank 1, we may consider

$$[\Omega_X].$$

This class is traditionally called the *canonical class of divisors* of X and denoted by K.

Remark 5.4.10. It follows from what precedes that

$$D \mapsto \mathcal{O}_X(D)$$

induces a bijection between $\mathrm{Pic}(X)$ and the set of isomorphy classes of locally free $\mathcal{O}_X$-module of rank 1 on X whose inverse is the map induced by

$$\mathcal{L} \mapsto [\mathcal{L}].$$

Since

$$\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D') \simeq \mathcal{O}_X(D + D'),$$

the preceding bijection is in fact an isomorphism of abelian groups.

As a consequence of this isomorphism we get that

$$\chi(X; \mathcal{L}) = \deg[\mathcal{L}] + 1 - g$$

for any locally free $\mathcal{O}_X$ module of rank 1 on X

Let us now look at Roch's original contribution.

Definition 5.4.11. We denote

$$\ell : \mathrm{Pic}(X) \to \mathbb{N}$$

the map induced by

$$D \mapsto \dim \mathrm{H}^0(X; \mathcal{O}_X(D)).$$

Example 5.4.12. We have

$$\ell(0) = \dim \mathrm{H}^0(X; \mathcal{O}_X) = 1$$

and

$$\ell(K) = \dim \mathrm{H}^0(X; \Omega_X) = g.$$

Proposition 5.4.13. *For any divisor D on X, we have*

$$\ell([D]) - \ell(K - [D]) = \deg D + 1 - g$$

Proof. From the duality theorem for coherent analytic sheaf we deduce that

$$\mathrm{H}^1(X; \mathcal{O}_X(D)) \simeq \mathrm{H}^0(X; \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D), \Omega_X)).$$

Using the isomorphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D), \Omega_X) \simeq \mathcal{O}_X(D)^* \otimes \Omega_X,$$

and the properties of $[\cdot]$, we see that

$$[\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D), \Omega_X)] = K - [D].$$

It follows that

$$\dim \mathrm{H}^1(X; \mathcal{O}_X(D)) = \ell(K - D).$$

Since

$$\chi(X; \mathcal{O}_X(D)) = \deg D + 1 - g,$$

the proof is complete. □

Corollary 5.4.14. *We have*

$$\deg K = 2g - 2.$$

Proof. Applying the preceding proposition for $D = K$, we obtain

$$\ell(K) - \ell(O) = \deg K + 1 - g.$$

Therefore,

$$g - 1 = \deg K + 1 - g$$

and the conclusion follows. □

Proposition 5.4.15.

(a) *If* $\deg D < 0$*, then*

$$\dim \mathrm{H}^0(X; \mathcal{O}_X(D)) = 0 \quad \text{and} \quad \dim \mathrm{H}^1(X; \mathcal{O}_X(D)) = g - 1 - \deg D.$$

(b) *If* $\deg D > 2g - 2$*, then*

$$\dim \mathrm{H}^0(X; \mathcal{O}_X(D)) = \deg D + 1 - g \quad \text{and} \quad \dim \mathrm{H}^1(X; \mathcal{O}_X(D)) = 0.$$

Proof. (a) Thanks to Riemann-Roch formula, it is sufficient to show that

$$\mathrm{H}^0(X; \mathcal{O}_X(D)) = 0.$$

Let us proceed by contradiction. Assume f is a non-zero section of $\mathcal{O}_X(D)$. Then,

$$(f) \geq -D$$

and

$$\deg(f) + \deg D \geq 0.$$

Since $[(f)] = 0$, we have $\deg(f) = 0$ and we get $\deg D \geq 0$. The conclusion follows.

(b) As in (a), we have only to prove the first equality. Using Proposition 5.4.13, we see that this equality is true if

$$\ell(K - [D]) = 0.$$

Since by the preceding corollary we have $\deg(K - [D]) = 2g - 2 - \deg(D)$, this follows from directly from part (a). ☐

Remark 5.4.16. Note that the second part of the preceding proposition contains a complete answer to the original Riemann-Roch problem for high degree divisors.

Proposition 5.4.17. *For any divisor D on X, we have*

$$\deg D = \int_X c_1(\mathcal{O}_X(D)).$$

Proof. Assume $D = \sum_{i=1}^m n_i[x_i]$ where $x_1, \ldots, x_m$ are distinct points of X. For any $i \in \{1, \ldots, m\}$, let t_i be local coordinate defined on an open neighborhood U_i of x_i such that $t_i(x_i) = 0$, $t_i(U_i) \simeq D(0,1)$ and assume that $U_1, \ldots, U_m$ are disjoint. Set $U_0 = X \setminus \{x_1, \ldots, x_m\}$ and denote $\mathcal{U}$ the covering $\{U_0, \ldots, U_m\}$. We know that $\mathcal{O}_X(D)$ is generated on $U_0, U_1, \ldots, U_m$ by the functions $1, t_1^{-n_1}, \ldots, t_m^{-n_m}$. So, we may represent the class of $\mathcal{O}_X(D)$ in $\mathrm{H}^1(X; \mathcal{O}_X^*) \simeq \check{\mathrm{H}}^1(\mathcal{U}, \mathcal{O}_X^*)$ by the Čech cocycle g defined by setting

$$g_{U_0U_1} = t_1^{n_1}, \quad g_{U_0U_2} = t_2^{n_2}, \ldots, g_{U_0U_m} = t_m^{n_m}.$$

Thanks to the proof of Proposition 4.2.10, we also know that the diagram

$$\begin{tikzcd} \mathrm{H}^1(X, \mathcal{O}_X^*) \arrow[r, "\mathrm{dlog}"] \arrow[dr, "c_1"'] & \mathrm{H}^1(X, \Omega^1) \arrow[d] \\ & \mathrm{H}^2(X, \mathbb{C}) \end{tikzcd}$$

is commutative. It follows that $c_1(\mathcal{O}_X(D))$ is the image in $\mathrm{H}^2(X; \mathbb{C})$ of the Čech cocycle of $\check{\mathrm{H}}^1(\mathcal{U}, \Omega^1)$ defined by

$$\begin{aligned} c^1_{U_0U_1} &= \mathrm{dlog}\, g_{U_0U_1} = n_1 \frac{dt_1}{t_1} \\ &\vdots \\ c^1_{U_0U_m} &= \mathrm{dlog}\, g_{U_0U_m} = n_m \frac{dt_m}{t_m}. \end{aligned}$$

The Dolbeault resolution

$$0 \to \Omega^1 \to \mathcal{C}_\infty^{(1,0)} \to \mathcal{C}_\infty^{(1,1)} \to 0$$

induces an isomorphism

$$\mathrm{H}^1(X, \Omega^1) \simeq \Gamma(X, \mathcal{C}_\infty^{(1,1)})/\overline{\partial}\Gamma(X; \mathcal{C}_\infty^{(1,0)}).$$

We may follow it in terms of Čech cohomology thanks to the Weil Lemma. We have to follow the dotted path in the diagram

$$\begin{tikzcd}
\Gamma(X;\Omega^1) \arrow[r] \arrow[d] & \Gamma(X;\mathcal{C}_\infty^{(1,0)}) \arrow[r, dashed] \arrow[d, dashed] & \Gamma(X;\mathcal{C}_\infty^{(1,1)}) \arrow[d] \\
\check{C}^0(\mathcal{U},\Omega^1) \arrow[r] \arrow[d] & \check{C}^0(\mathcal{U},\mathcal{C}_\infty^{(1,0)}) \arrow[r] \arrow[d, dashed] & \check{C}^0(\mathcal{U},\mathcal{C}_\infty^{(1,1)}) \arrow[d] \\
\check{C}^1(\mathcal{U},\Omega^1) \arrow[r] \arrow[d] & \check{C}^1(\mathcal{U},\mathcal{C}_\infty^{(1,0)}) \arrow[r] \arrow[d] & \check{C}^1(\mathcal{U},\mathcal{C}_\infty^{(1,1)}) \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

To write c^1 as the Čech coboundary of $c^0 \in \check{C}^0(\mathcal{U}, \mathcal{C}_\infty^{(1,0)})$, we will use a partition unity $(\varphi_U)_{U\in\mathcal{U}}$ relative to $\mathcal{U}$ and set

$$\begin{aligned}
c^0_{U_0} &= -\varphi_{U_1} c^1_{U_0U_1} - \cdots - \varphi_{U_m} c^1_{U_0U_m} \\
c^0_{U_1} &= \varphi_{U_0} c^1_{U_0U_1} \\
&\vdots \\
c^0_{U_m} &= \varphi_{U_0} c^1_{U_0U_m}
\end{aligned}$$

Then, we may represent the image of c in $\mathrm{H}^2(X, \mathbb{C})$ by the Čech 0-cocycle of $\check{C}^0(\mathcal{U}, \mathcal{C}_\infty^{(1,1)})$ given by

$$\begin{aligned}
U_0 &\mapsto dc^0_{U_0} \\
&\vdots \\
U_m &\mapsto dc^0_{U_m}
\end{aligned}$$

which corresponds to the differential form with compact support

$$\omega = -d\varphi_{U_1} \wedge n_1 \frac{dt_1}{t_1} - \cdots - d\varphi_{U_m} \wedge n_m \frac{dt_m}{t_m}.$$

It follows that

$$\int_X c_1(\mathcal{O}_X(D)) = \int_X \omega = -\sum_{i=1}^m n_i \int_{U_i} d\varphi_{U_i} \wedge \frac{dt_i}{t_i}.$$

Denote V_i the image in U_i of a disk centered on 0 whose radius is sufficiently small in order that $\varphi_{U_i} = 1$ on $\overline{V}_i \subset U_i$. Stokes' theorem shows that

$$\begin{aligned}\int_X c_1(\mathcal{O}_X(D)) &= -\sum_{i=1}^m n_i \int_{U_i \setminus V_i} d(\varphi_{U_i} \frac{dt_i}{t_i}) \\ &= \sum_{i=1}^m n_i \int_{\partial V_i} \varphi_{U_i} \frac{dt_i}{t_i} \\ &= \sum_{i=1}^m n_i \int_{\partial V_i} \frac{dt_i}{t_i} = \deg D.\end{aligned}$$

□

Corollary 5.4.18. *For any locally free $\mathcal{O}_X$-module $\mathcal{L}$ of rank 1 on X, we have*

$$\chi(X; \mathcal{L}) = \int_X c_1(\mathcal{L}) + \frac{c_1(TX)}{2}.$$

Proof. Since TX is a holomorphic line bundle, we know that $c_1(TX)$ is the Euler class of TX and hence of X. Therefore,

$$\int_X c_1(TX) = \chi(X, \mathbb{C}) = 1 - 2g + 1 = 2(1-g)$$

and

$$1 - g = \int_X \frac{c_1(TX)}{2}.$$

Since by the preceding proposition, we have

$$\deg([\mathcal{L}]) = \int_X c_1(\mathcal{L}),$$

Remark 5.4.10 allows us to conclude. □

Remark 5.4.19. Thanks to the computations at the end of Section 1, the preceding corollary shows that Hirzebruch-Riemann-Roch formula is true for compact complex curves.

5.5 Cohomology of coherent analytic sheaves on $\mathbb{P}_n(\mathbb{C})$

Definition 5.5.1. Let

$$q : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}_n(\mathbb{C})$$

be the canonical projection. For any open subset U of $\mathbb{P}_n(\mathbb{C})$ and any $l \in \mathbb{Z}$, set

$$\mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l)(U) = \{f \in \mathcal{O}_{\mathbb{C}^{n+1}}(q^{-1}(U)) : f(\lambda z) = \lambda^l f(z) \ (\lambda \in \mathbb{C}^*)\}.$$

From this definition it follows immediately that :

Proposition 5.5.2. *For any $l \in \mathbb{Z}$,*

$$U \mapsto \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l)(U)$$

is a locally free $\mathcal{O}_{\mathbb{P}_n(\mathbb{C})}$-module of rank 1. Moreover, there are canonical isomorphisms

$$\mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l) \otimes_{\mathcal{O}_{\mathbb{P}_n(\mathbb{C})}} \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l') \simeq \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l+l'),$$

$$\mathcal{H}om_{\mathcal{O}_{\mathbb{P}_n(\mathbb{C})}}(\mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l), \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l')) \simeq \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l'-l).$$

Exercise 5.5.3. *There is a canonical isomorphism*

$$\Omega_{\mathbb{P}_n(\mathbb{C})} \simeq \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(-n-1).$$

Solution. As usual, set

$$U_k = \{[z_0, \cdots, z_n] : z_k \neq 0\}$$

and define the map $u_{kl} : U_k \to \mathbb{C}$ by setting

$$u_{kl} = \frac{z_l}{z_k}.$$

Then, $(u_{k0}, \cdots, \widehat{u_{kk}}, \cdots, u_{kn})$ gives a biholomorphic bijection between U_k and $\mathbb{C}^n$. It follows that any $\omega \in \Gamma(U; \Omega_{\mathbb{P}_n(\mathbb{C})})$ may be written in a unique way on $U \cap U_k$ as

$$\omega = f_k du_{k0} \wedge \cdots \wedge \widehat{du_{kk}} \wedge \cdots \wedge du_{kn}$$

with $f_k \in \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(U \cap U_k)$. For $l < k$, we have

$$u_{km} = \frac{u_{lm}}{u_{lk}}.$$

Therefore,

$$du_{km} = \frac{u_{lk} du_{lm} - u_{lm} du_{lk}}{u_{lk}^2}$$

and

$$\begin{aligned}
&du_{k0} \wedge \cdots \wedge \widehat{du_{kk}} \wedge \cdots \wedge du_{kn} \\
&\quad = \sum_{m=0}^{k-1} \frac{du_{l0}}{u_{lk}} \wedge \cdots \wedge \frac{-u_{lm} du_{lk}}{u_{lk}^2} \wedge \cdots \wedge \frac{\widehat{du_{lk}}}{u_{lk}} \wedge \cdots \wedge \frac{du_{ln}}{u_{lk}} \\
&\qquad + \sum_{m=k+1}^{n} \frac{du_{l0}}{u_{lk}} \wedge \cdots \wedge \frac{\widehat{du_{lk}}}{u_{lk}} \wedge \cdots \wedge \frac{-u_{lm} du_{lk}}{u_{lk}^2} \wedge \cdots \wedge \frac{du_{ln}}{u_{lk}}.
\end{aligned}$$

Since $du_{ll} = 0$, we get

$$\begin{aligned} &du_{k_0} \wedge \cdots \wedge \widehat{du_{kk}} \wedge \cdots \wedge du_{kn} \\ &\quad = \frac{du_{l0}}{u_{lk}} \wedge \cdots \wedge \frac{-u_{ll} du_{lk}}{u_{lk}^2} \wedge \cdots \wedge \frac{\widehat{du_{lk}}}{u_{lk}} \wedge \cdots \wedge \frac{du_{ln}}{u_{lk}} \\ &\quad = \frac{(-1)^{k-l}}{u_{lk}^{n+1}} du_{l_0} \wedge \cdots \wedge \widehat{du_{ll}} \wedge \cdots \wedge du_{ln}. \end{aligned}$$

It follows that

$$f_l = f_k \frac{(-1)^{k-l}}{u_{lk}^{n+1}}.$$

Hence,

$$\frac{(-1)^l}{z_l^{n+1}} f_l([z_0, \cdots, z_n]) = \frac{(-1)^k}{z_k^{n+1}} f_k([z_0, \cdots, z_n])$$

on $q^{-1}(U \cap U_k \cap U_l)$. This shows that there is a unique $h \in \Gamma(U; \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(-n-1))$ such that

$$h(z_0 \cdots, z_n) = \frac{(-1)^k}{z_k^{n+1}} f_k([z_0, \cdots, z_n])$$

on $q^{-1}(U \cap U_k)$ for $k \in \{0, \cdots, n\}$. We have thus constructed a canonical morphism

$$\Omega_{\mathbb{P}_n(\mathbb{C})} \to \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(-n-1).$$

Since it is clearly an isomorphism, the proof is complete. □

Exercise 5.5.4. *Denote $\mathcal{U}_n(\mathbb{C})$ the sheaf of holomorphic sections of the universal bundle $\mathbb{U}_n(\mathbb{C})$ on $\mathbb{P}_n(\mathbb{C})$. Then, there is a canonical isomorphism*

$$\mathcal{U}_n(\mathbb{C}) \simeq \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(-1).$$

Solution. Using the same notations as in the proof of the preceding proposition, set

$$s_k([z_0, \cdots, z_n]) = \left((\frac{z_0}{z_k}, \cdots, \frac{\widehat{z_k}}{z_k}, \cdots, \frac{z_n}{z_k}), [z_0, \cdots, z_n] \right).$$

Clearly, s_k is a holomorphic frame of $\mathbb{U}_n(\mathbb{C})_{|U_k}$. Therefore, for any $\sigma \in \Gamma(U; \mathcal{U}_n(\mathbb{C}))$ there is a unique $\sigma_k \in \Gamma(U \cap U_k; \mathcal{O}_{\mathbb{P}_n(\mathbb{C})})$ such that

$$\sigma = \sigma_k s_k$$

on $U \cap U_k$. Since

$$s_k = u_{kl} s_l$$

on $U_k \cap U_l$, we have

$$\sigma_l = \sigma_k u_{kl}$$

on $U \cap U_k \cap U_l$. It follows that

$$\frac{\sigma_l([z_0, \cdots, z_n])}{z_l} = \frac{\sigma_k([z_0, \cdots, z_n])}{z_k}$$

on $q^{-1}(U \cap U_k \cap U_l)$ and hence that there is a unique $h \in \Gamma(U; \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(-1))$ such that

$$h(z_0, \cdots, z_n) = \frac{\sigma_k([z_0, \cdots, z_n])}{z_k}$$

on $q^{-1}(U \cap U_k)$. The conclusion follows as in the preceding proposition. $\square$

Theorem 5.5.5. *The cohomology of the sheaf $\mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l)$ is given by the table :*

$$\mathrm{H}^0(\mathbb{P}_n(\mathbb{C}); \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l)) = \begin{cases} \mathbb{C}^{\binom{n+l}{l}} & \textit{if } l \geq 0 \\ 0 & \textit{otherwise} \end{cases}$$

$$\mathrm{H}^k(\mathbb{P}_n(\mathbb{C}); \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l)) = 0 \quad \textit{if } 0 < k < n$$

$$\mathrm{H}^n(\mathbb{P}_n(\mathbb{C}); \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l)) = \begin{cases} \mathbb{C}^{\binom{-l-1}{n}} & \textit{if } l \leq -n-1 \\ 0 & \textit{otherwise} \end{cases}$$

In particular,

$$\chi(\mathbb{P}_n(\mathbb{C}); \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l)) = \frac{(l+1)\cdots(l+n)}{n!}$$

for any $l \in \mathbb{Z}$.

Proof. We will only do the easy part and prove the first equality. We refer to [26] and [27] for the other results. Note that it follows from definitions that

$$\mathrm{H}^0(\mathbb{P}_n(\mathbb{C}); \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l)) = \{f \in \mathcal{O}_{\mathbb{C}^{n+1}}(\mathbb{C}^{n+1} \setminus \{0\}) : f(\lambda z) = \lambda^l f(z) \ (\lambda \in \mathbb{C}^*)\}.$$

Since the codimension of $\{0\}$ in $\mathbb{C}^{n+1}$ is at least 2, any

$$f \in \mathrm{H}^0(\mathbb{P}_n(\mathbb{C}); \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l))$$

is in fact holomorphic on $\mathbb{C}^{n+1}$ and may thus be written as

$$f(z) = \sum_{\alpha_0=0}^{\infty} \cdots \sum_{\alpha_n=0}^{\infty} a_{\alpha_0, \cdots, \alpha_n} z_0^{\alpha_0} \cdots z_n^{\alpha_n}$$

with $a_{\alpha_0,\cdots,\alpha_n} \in \mathbb{C}$. From the relation

$$f(\lambda z) = \lambda^l f(z) \qquad (\lambda \in \mathbb{C}^*),$$

we deduce that $a_{\alpha_0,\cdots,\alpha_n}\lambda^{|\alpha|} = a_{\alpha_0,\cdots,\alpha_n}\lambda^l$ and hence that $a_{\alpha_0,\cdots,\alpha_n} = 0$ if $|\alpha| \neq l$. It follows that $\mathrm{H}^0(\mathbb{P}_n(\mathbb{C}); \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l))$ is 0 if $l < 0$ and equal to the space of homogeneous polynomials of degree l for $l \geq 0$. In this case, the dimension is given by

$$d_{n,l} = \#\{(\alpha_0, \cdots, \alpha_n) : \alpha_0 \geq 0, \cdots, \alpha_n \geq 0, \alpha_0 + \cdots + \alpha_n = l\}.$$

To compute this number, we note that, since

$$\left(\sum_{\alpha_0=0}^{\infty} z^{\alpha_0}\right) \cdots \left(\sum_{\alpha_n=0}^{\infty} z^{\alpha_n}\right) = \sum_{\alpha_0=0}^{\infty} \cdots \sum_{\alpha_n=0}^{\infty} z^{\alpha_0+\cdots+\alpha_n},$$

we have

$$\frac{1}{(1-z)^{n+1}} = \sum_{l=0}^{\infty} d_{n,l} z^l.$$

Hence,

$$d_{n,l} = \frac{1}{l!}\left[\frac{1}{(1-z)^{n+1}}\right]^{(l)}_{z=0} = \frac{(n+1)(n+2)\cdots(n+l)}{l!} = \binom{n+l}{l}.$$

To conclude, note that it follows from the cohomology table that

$$\chi(\mathbb{P}_n(\mathbb{C}); \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l)) = \begin{cases} \binom{n+l}{l} & \text{if } l \geq 0 \\ 0 & \text{if } -n \leq l \leq -1 \\ (-1)^n \binom{-l-1}{n} & \text{if } l \leq -n-1 \end{cases}$$

and the announced formula holds since for $l \leq -n-1$, we have

$$(-1)^n \frac{(-l-n)\cdots(-l-1)}{n!} = \frac{(l+1)\cdots(l+n)}{n!}.$$

□

Remark 5.5.6. Note that, using the duality theorem for coherent analytic sheaves, we have

$$\begin{aligned} \mathrm{H}^k(\mathbb{P}_n(\mathbb{C}); \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l)) &\simeq \mathrm{H}^{n-k}(\mathbb{P}_n(\mathbb{C}); \mathcal{H}om_{\mathcal{O}_{\mathbb{P}_n(\mathbb{C})}}(\mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l), \Omega_{\mathbb{P}_n(\mathbb{C})})) \\ &\simeq \mathrm{H}^{n-k}(\mathbb{P}_n(\mathbb{C}); \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(-l-1-n)). \end{aligned}$$

Therefore, the last isomorphism in the preceding theorem follows from the first since

$$\binom{n-l-1-n}{-l-1-n} = \binom{-l-1}{n}$$

if $l \leq -n-1$.

Corollary 5.5.7. *The graded* $\mathbb{C}$*-algebra*

$$S = \bigoplus_{l \in \mathbb{Z}} \Gamma(\mathbb{P}_n(\mathbb{C}); \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l))$$

is canonically isomorphic to $\mathbb{C}[z_0, \cdots, z_n]$.

Definition 5.5.8. Recall that the homogeneous part of degree l of a graded S-module M is denoted M_l and that $M(l)$ is the graded S-module characterized by the fact that $M(l)_{l'} = M_{l+l'}$ $(l' \in \mathbb{Z})$.

We denote $\mathcal{S}$ the sheaf of graded $\mathbb{C}$-algebras defined by setting

$$\mathcal{S} = \bigoplus_{l \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l).$$

For any graded S-module M, consider the graded tensor product

$$\mathcal{S} \otimes_S M.$$

This is clearly a graded $\mathcal{S}$-module. We denote $\tilde{M}$ its homogeneous part of degree 0. By construction, $\tilde{M}$ is an $\mathcal{O}_{\mathbb{P}_n(\mathbb{C})}$-module. Finally, we denote $\mathcal{PN}$ (resp. $\mathcal{PF}$) the thick subcategory of $\mathcal{M}\mathrm{od}(S)$ formed by the graded S-modules M such that there is $l_0 \in \mathbb{Z}$ with $\bigoplus_{l \geq l_0} M_l$ isomorphic to zero (resp. of finite type).

Proposition 5.5.9. *The functor*

$$M \to \tilde{M}$$

from the category of graded S*-modules to that of* $\mathcal{O}_{\mathbb{P}_n(\mathbb{C})}$*-modules is exact. Moreover,*

(a) $\tilde{S}(l) = \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l)$,

(b) $\tilde{M} = 0$ *if* M *is an object of* $\mathcal{PN}$,

(c) $\tilde{M}$ *is coherent if* M *is an object of* $\mathcal{PF}$.

Proof. The difficult part is the exactness, for it we refer the reader to [26] and [27]; the other parts are easier. As a matter of fact, (b) follows at once from the fact that for $x \in U_k$, any section of $(\tilde{M})_x$ of the form

$$\sigma_{l_0} \otimes m_{-l_0}$$

with $\sigma_{l_0} \in (\mathcal{S}_{l_0})_x$, $m_{-l_0} \in M_{-l_0}$ is equal to

$$\sigma_{l_0} z_k^{l_0 - l} \otimes z_k^{l - l_0} m_{-l_0}$$

for any $l \in \mathbb{Z}$. As for (c), we know that there is a morphism

$$\bigoplus_{r_0=1}^{R_0} S(l_{r_0}) \to M$$

with a cokernel in $\mathcal{PN}$. Since S is noetherian, we get an exact sequence of the type

$$\bigoplus_{r_1=1}^{R_1} S(l_{r_1}) \to \bigoplus_{r_0=1}^{R_0} S(l_{r_0}) \to M \to N \to 0$$

with $N \in \mathcal{PN}$. This gives us the exact sequence of $\mathcal{O}_{\mathbb{P}_n(\mathbb{C})}$-modules

$$\bigoplus_{r_1=1}^{R_1} \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l_{r_1}) \to \bigoplus_{r_0=1}^{R_0} \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l_{r_0}) \to \tilde{M} \to 0$$

and the conclusion follows. □

Definition 5.5.10. Let $\mathcal{F}$ be a coherent analytic sheaf on $\mathbb{P}_n(\mathbb{C})$. We set

$$\mathcal{F}(l) = \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l) \otimes_{\mathcal{O}_{\mathbb{P}_n(\mathbb{C})}} \mathcal{F}$$

and denote $S(\mathcal{F})$ the graded S-module

$$\bigoplus_{l\in\mathbb{Z}} \Gamma(\mathbb{P}_n(\mathbb{C}); \mathcal{F}(l)).$$

Proposition 5.5.11. *For any coherent analytic sheaf $\mathcal{F}$ on $\mathbb{P}_n(\mathbb{C})$, the graded S-module $S(\mathcal{F})$ is an object of $\mathcal{PF}$. Moreover, the functor*

$$\mathcal{F} \mapsto S(\mathcal{F})$$

induces an equivalence between the category of coherent analytic sheaves on $\mathbb{P}_n(\mathbb{C})$ and the abelian category $\mathcal{PF}/\mathcal{PN}$. The quasi-inverse of this equivalence is induced by the functor

$$M \mapsto \tilde{M}.$$

Proof. We refer the reader to [26] and [27]. □

Corollary 5.5.12. *Let $\mathcal{F}$ be a coherent analytic sheaf on $\mathbb{P}_n(\mathbb{C})$. Then,*

(a) *there is an exact sequence of the form*

$$0 \to \bigoplus_{r_p=0}^{R_p} \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l_{r_p}) \to \cdots \to \bigoplus_{r_0=0}^{R_0} \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l_{r_0}) \to \mathcal{F} \to 0;$$

(b) *the $\mathcal{O}_{\mathbb{P}_n(\mathbb{C})}$-module $\mathcal{F}(l)$ is generated by its global sections for $l \gg 0$;*

(c) *we have*

$$\mathrm{H}^k(\mathbb{P}_n(\mathbb{C}); \mathcal{F}(l)) = 0 \qquad (k > 0)$$

for $l \gg 0$.

Proof. (a) We know that $S(\mathcal{F})$ is an object of $\mathcal{PF}$. It follows that there is a morphism

$$\bigoplus_{r_0=0}^{R_0} S(l_{r_0}) \to S(\mathcal{F})$$

with a cokernel N in $\mathcal{PN}$. Since its kernel K is a graded submodule of a graded S-module of finite type, Hilbert syzygies theorem shows that K has a presentation of the form

$$0 \to \bigoplus_{r_p=0}^{R_p} S(l_{r_p}) \to \cdots \to \bigoplus_{r_1=0}^{R_1} S(l_{r_1}) \to K \to 0.$$

It follows that we have an exact sequence of the form

$$0 \to \bigoplus_{r_p=0}^{R_p} S(l_{r_p}) \to \cdots \to \bigoplus_{r_0=0}^{R_0} S(l_{r_0}) \to S(\mathcal{F}) \to N \to 0.$$

The conclusion follows by applying the exact functor $\tilde{\cdot}$ to this sequence.

(b) follows from (a) since the result is true for $\mathcal{F} = \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l)$ with $l \geq 0$.

(c) follows directly from (a) thanks to Theorem 5.5.5. $\square$

Remark 5.5.13. Parts (b) and (c) of the preceding proposition may be viewed as stating that $\mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(1)$ is an ample locally free $\mathcal{O}_{\mathbb{P}_n(\mathbb{C})}$-module of rank 1.

Proposition 5.5.14. *Let M be a graded S-module which is in $\mathcal{PF}$. Then, there is a unique polynomial $P_M \in \mathbb{Q}[z]$ such that*

$$P_M(l) = \dim M_l$$

for $l \gg 0$.

Proof. We know that for M in $\mathcal{PF}$ there is an exact sequence of the type

$$0 \to \bigoplus_{r_p=0}^{R_p} S(l_{r_p}) \to \cdots \to \bigoplus_{r_0=0}^{R_0} S(l_{r_0}) \to M \to N \to 0$$

with N in $\mathcal{PN}$. Moreover,

$$\dim S(l_0)_l = \dim S_{l_0+l} = \frac{(l+l_0+1)\cdots(l+l_0+n)}{n!}.$$

Hence, the conclusion follows from the additivity of $M \mapsto \dim M_l$ and the fact that the zeros of a polynomial are in finite number. ☐

Definition 5.5.15. The polynomial P_M of Proposition 5.5.14 is called the *Hilbert-Samuel polynomial* of M.

Corollary 5.5.16. *For any coherent analytic sheaf $\mathcal{F}$ on $\mathbb{P}_n(\mathbb{C})$ the map*

$$l \mapsto \chi(\mathbb{P}_n(\mathbb{C}); \mathcal{F}(l))$$

is the Hilbert-Samuel polynomial $P_{S(\mathcal{F})}$ of the graded S-module $S(\mathcal{F})$ (in other words

$$\chi(\mathbb{P}_n(\mathbb{C}); \mathcal{F}(l)) = \dim \Gamma(\mathbb{P}_n(\mathbb{C}); \mathcal{F}(l))$$

for $l \gg 0$).

Proof. It follows from Theorem 5.5.5 that

$$\chi(\mathbb{P}_n(\mathbb{C}); \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l_0)(l)) = \frac{(l+l_0+1)\cdots(l+l_0+n)}{n!}.$$

Therefore,

$$\chi(\mathbb{P}_n(\mathbb{C}); \mathcal{F}(l))$$

is a polynomial in l for $\mathcal{F} = \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(l_0)$. Part (a) of Corollary 5.5.12 and the additivity of $\chi(\mathbb{P}_n(\mathbb{C}); \cdot)$ allow us to extend the result to any coherent analytic sheaf $\mathcal{F}$ on $\mathbb{P}_n(\mathbb{C})$. Moreover, using part (c) of Corollary 5.5.12, we see that

$$\chi(\mathbb{P}_n(\mathbb{C}); \mathcal{F}(l)) = \dim \Gamma(\mathbb{P}_n(\mathbb{C}); \mathcal{F}(l))$$

if $l \gg 0$ and the conclusion follows from the definition of $S(\mathcal{F})$ and of its Hilbert-Samuel polynomial. ☐

Exercise 5.5.17. *Let Z be a closed hypersurface of $\mathbb{P}_n(\mathbb{C})$ corresponding to the zeros of a homogeneous polynomial Q of degree d. Denote $\mathcal{O}_Z$ the coherent analytic sheaf on $\mathbb{P}_n(\mathbb{C})$ associated to Z. Show that*

$$\begin{aligned} &P_{S/(Q)}(l) \\ &\qquad = \chi(\mathbb{P}_n(\mathbb{C}); \mathcal{O}_Z(l)) \\ &\qquad = \frac{(l+1)\cdots(l+n)}{n!} - \frac{(l-d+1)\cdots(l-d+n)}{n!} \end{aligned}$$

and that in particular

$$\chi(\mathbb{P}_n(\mathbb{C}); \mathcal{O}_Z) = \begin{cases} 1 & \text{if } d \leq n \\ 1 - (-1)^n \binom{d-1}{n} & \text{if } d > n. \end{cases}$$

Solution. From the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(-d) \xrightarrow{Q} \mathcal{O}_{\mathbb{P}_n(\mathbb{C})} \to \mathcal{O}_Z \to 0$$

we deduce that

$$\chi(\mathbb{P}_n(\mathbb{C}); \mathcal{O}_Z) = \chi(\mathbb{P}_n(\mathbb{C}); \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}) - \chi(\mathbb{P}_n(\mathbb{C}); \mathcal{O}_{\mathbb{P}_n(\mathbb{C})}(-d))$$

and the first formula follows immediately. For the second one, we note that

$$\frac{(-d+1)\cdots(-d+n)}{n!} = \begin{cases} 0 & \text{if } d \leq n \\ (-1)^n \dfrac{(d-1)\cdots(d-n)}{n!} & \text{if } d > n. \end{cases}$$

Hence, the conclusion. □

5.6 Hirzebruch-Riemann-Roch theorem for $\mathbb{P}_n(\mathbb{C})$

Proposition 5.6.1. *Assume $\mathcal{F}$ is a coherent analytic sheaf on $X = \mathbb{P}_n(\mathbb{C})$. Then,*

$$\chi(X; \mathcal{F}) = \int_X \operatorname{ch}(\mathcal{F}) \smile \operatorname{td}(TX).$$

Proof. By Corollary 5.5.12, we know that $\mathcal{F}$ has a resolution of the form

$$0 \to \bigoplus_{r_p=0}^{R_p} \mathcal{O}_X(l_{r_p}) \to \cdots \to \bigoplus_{r_0=0}^{R_0} \mathcal{O}_X(l_{r_0}) \to \mathcal{F} \to 0.$$

Using the additivity of both sides of Hirzebruch-Riemann-Roch formula, it is thus sufficient to treat the case $\mathcal{F} = \mathcal{O}_X(l)$ ($l \in \mathbb{Z}$). Thanks to Exercise 4.2.6 we know that $c.(TX) = (1+\xi)^{n+1}$ where $\xi \in \mathrm{H}^2(X; \mathbb{Z})$ is the first Chern class of $\mathbb{U}_n^*(\mathbb{C})$. Hence, TX and $\mathbb{U}_n^*(\mathbb{C})^{n+1}$ have same total Chern class. It follows that

$$\operatorname{td}(TX) = \operatorname{td}(\mathbb{U}_n^*(\mathbb{C})^{n+1}) = (\operatorname{td} \mathbb{U}_n^*(\mathbb{C}))^{n+1} = \left(\frac{\xi}{1-e^{-\xi}}\right)^{n+1}.$$

Recall that the sheaf of holomorphic sections of $\mathbb{U}_n(\mathbb{C})$ is $\mathcal{O}_X(-1)$. Hence, for $l > 0$, we have

$$\begin{aligned} \operatorname{ch}(\mathcal{O}_X(l)) &= \operatorname{ch}(\mathcal{O}_X(1) \otimes \cdots \otimes \mathcal{O}_X(1)) \\ &= \operatorname{ch}(\mathcal{O}_X(1))^l = e^{l\xi} \end{aligned} \tag{*}$$

Using the fact that

$$\mathrm{ch}(\mathcal{O}_X(-l))\,\mathrm{ch}(\mathcal{O}_X(l)) = \mathrm{ch}(\mathcal{O}_X) = 1$$

we see that formula (*) holds for any $l \in \mathbb{Z}$. Combining the results, we get that

$$\mathrm{ch}(\mathcal{O}_X(l))\,\mathrm{td}(TX) = \left(\frac{\xi}{1-e^{-\xi}}\right)^{n+1} e^{l\xi}.$$

Therefore,

$$\left(\mathrm{ch}(\mathcal{O}_X(l))\,\mathrm{td}(TX)\right)^{2n} = \rho\xi^n$$

where ρ is the coefficient of z^n in the Taylor expression at 0 of

$$\left(\frac{z}{1-e^{-z}}\right)^{n+1} e^{lz}.$$

Using Cauchy's formula, we have

$$\rho = \mathrm{Res}\left(\frac{e^{lz}}{(1-e^{-z})^{n+1}};0\right) = \frac{1}{2i\pi}\int_C \frac{e^{lz}}{(1-e^{-z})^{n+1}}dz$$

where C is a path of $\mathbb{C}^*$ such that $\mathrm{Ind}(\gamma,0) = 1$. Using the change of variables

$$w = 1 - e^{-z}$$

we get

$$\begin{aligned}
\rho &= \frac{1}{2i\pi}\int_C \frac{(1-w)^{-l}}{w^{n+1}}\frac{dw}{1-w} \\
&= \frac{1}{2i\pi}\int_C \frac{dw}{w^{n+1}(1-w)^{l+1}} \\
&= \frac{1}{n!}\left(\frac{1}{(1-w)^{l+1}}\right)^{(n)}_{w=0} = \frac{(l+1)\cdots(l+n)}{n!} = \binom{l+n}{n}.
\end{aligned}$$

By Exercise 4.2.6, we know that $\int_X \xi^n = 1$. Therefore,

$$\int_X \mathrm{ch}(\mathcal{O}_X(l))\,\mathrm{td}(TX) = \binom{l+n}{n}$$

and the conclusion follows from Theorem 5.5.5. ☐

5.7 Riemann-Roch for holomorphic embeddings

Our aim in this section is to prove the following result.

Theorem 5.7.1. *Let $i : X \to Y$ be a closed embedding of complex analytic manifolds. Then, for any $\mathcal{F}^{\cdot} \in \mathcal{D}^{\mathrm{b}}_{\mathrm{Coh}}(\mathcal{O}_X)$, we have*

$$\mathrm{ch}_X(i_!\mathcal{F}) = i_!(\mathrm{ch}\,\mathcal{F}^{\cdot} / \mathrm{td}\,T_X Y)$$

in $\mathrm{H}^{\cdot}_X(Y;\mathbb{Q})$.

We will start by recalling a few facts about Koszul complexes which will be needed in the proof.

Let $\mathcal{R}$ be a commutative ring on the topological space X, let $\mathcal{E}$ be locally free $\mathcal{R}$-module of rank r and let s be a section of $\mathcal{E}^*$. Recall that the Koszul complex $K_{\cdot}(\mathcal{E}; s)$ is the complex

$$0 \to \bigwedge^r \mathcal{E} \xrightarrow{L_s} \cdots \to \bigwedge^k \mathcal{E} \xrightarrow{L_s} \cdots \to \mathcal{E} \xrightarrow{\partial_1} \mathcal{R} \to 0$$

where L_s is the interior product with s. As is well-known, the $\mathcal{R}$-linear morphism L_s is characterized by the formula

$$L_s(\sigma_1 \wedge \cdots \wedge \sigma_k) = \sum_{l=1}^{k} (-1)^{l-1} \langle s, \sigma_l \rangle \, \sigma_1 \wedge \cdots \wedge \widehat{\sigma_l} \wedge \cdots \wedge \sigma_k .$$

As a graded $\mathcal{R}$-module, $K_{\cdot}(\mathcal{E}; s)$ is isomorphic to the exterior algebra

$$\bigwedge^{\cdot} \mathcal{E}$$

and hence has a canonical structure of anticommutative graded $\mathcal{R}$-algebra. Note that the differential L_s is compatible with this structure in the sense that

$$L_s(\omega_k \wedge \omega_l) = L_s(\omega_k) \wedge \omega_l + (-1)^k \omega_k \wedge L_s(\omega_l)$$

for any sections $\omega_k \in \bigwedge^k \mathcal{E}$, $\omega_l \in \bigwedge^l \mathcal{E}$.

In particular, if $\mu_1, \ldots, \mu_p$ are global sections of $\mathcal{R}$, we have

$$K_{\cdot}(\mathcal{R}^p; (\mu_1, \cdots, \mu_p)) \simeq K_{\cdot}(\mathcal{R}; \mu_1) \otimes_{\mathcal{R}} \cdots \otimes_{\mathcal{R}} K_{\cdot}(\mathcal{R}; \mu_p).$$

From this formula, it follows easily that if

$$\mu_l : \mathcal{R}/\mathcal{R}\mu_1 + \cdots + \mathcal{R}\mu_{l-1} \to \mathcal{R}/\mathcal{R}\mu_1 + \cdots + \mathcal{R}\mu_{l-1}$$

is injective for $l \in \{1, \cdots, p\}$ then $K_{\cdot}(\mathcal{R}^p; (\mu_1, \cdots, \mu_p))$ is a projective resolution of

$$\mathcal{R}/\mathcal{R}\mu_1 + \cdots + \mathcal{R}\mu_p .$$

This allows us to prove the following lemma.

Lemma 5.7.2. *Let X be a complex analytic manifold and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank p. Denote E the associated holomorphic complex vector bundle. Assume s is a section of E which is transverse to the zero section. Then,*

$$Z_s = \{x \in X : s(x) = 0\}$$

is a complex analytic submanifold of X. Moreover, if $i : Z_s \to X$ is the inclusion map, then the Koszul complex

$$0 \to \bigwedge^{p} \mathcal{E}^* \xrightarrow{L_s} \bigwedge^{p-1} \mathcal{E}^* \to \cdots \to \mathcal{E}^* \xrightarrow{L_s} \mathcal{O}_x \to 0$$

is a resolution of the $\mathcal{O}_X$-module

$$i_! \mathcal{O}_{Z_s}.$$

Proof. The problem being clearly local on X, we may assume X is a coordinated neighborhood U of x_0 and $\mathcal{E} = \mathcal{O}_X^p$. Then, $E = \mathbb{C}^p \times U$ and we have

$$s(x) = (f(x), x)$$

where $f : U \to \mathbb{C}^p$ is a holomorphic map. The fact that s is transverse to the zero section entails that f is a submersion. So, restricting U to a smaller neighborhood of x_0 if necessary, we may find a holomorphic coordinate system $(z_1, \cdots, z_p)$ on U with $f_1 = z_1, \cdots, f_p = z_p$. For such a coordinate system, the Koszul complex

$$K_\cdot(\mathcal{E}, s) \simeq K_\cdot(\mathcal{O}_X^p, (z_1, \cdots, z_p))$$

and the conclusion follows easily. □

With this lemma at hand, we can now prove the following special case of Theorem 5.7.1.

Proposition 5.7.3. *Let $p : E \to X$ be a holomorphic complex vector bundle of rank r. Denote $i : X \to E$ its zero section. Then, for any $\mathcal{F}^\cdot \in \mathcal{D}^{\mathrm{b}}_{\mathcal{C}\mathrm{oh}}(\mathcal{O}_X)$, we have*

$$\mathrm{ch}_X(i_! \mathcal{F}^\cdot) = i_!(\mathrm{ch}\, \mathcal{F}^\cdot / \operatorname{td} E)$$

in $\mathrm{H}^\cdot_X(E; \mathbb{Q})$.

Proof. First, note that since $p \circ i = \mathrm{id}_X$, we have

$$\begin{aligned} \mathrm{ch}_X(i_! \mathcal{F}^\cdot) &= \mathrm{ch}_X(i_! i^* p^* \mathcal{F}^\cdot) \\ &= \mathrm{ch}_X(i_! \mathcal{O}_X \otimes^L{}_{\mathcal{O}_E} p^* \mathcal{F}^\cdot) \\ &= \mathrm{ch}_X(i_! \mathcal{O}_X) p^* \,\mathrm{ch}\, \mathcal{F}^\cdot \end{aligned}$$

and the result will be true for any $\mathcal{F}^{\cdot} \in \mathcal{D}^{\mathrm{b}}_{\mathcal{C}\mathrm{oh}}(\mathcal{O}_X)$ if it is true for $\mathcal{O}_X$.

To prove the result in this case, let us consider the relative projective compactification $\overline{p} : \overline{E} \to X$ of E. Recall that

$$\overline{E} = P(E \oplus (\mathbb{C} \times X))$$

and that we have an open embedding

$$j : E \to \overline{E}$$

and a complementary closed embedding

$$k : P(E) \to \overline{E}$$

which we can use to identify E and $P(E)$ with subspaces of $\overline{E}$. Set $\overline{i} = j \circ i$.

Let us show that

(a) the canonical morphism

$$\epsilon : \mathrm{H}^{\cdot}_X(\overline{E}; \mathbb{Q}) \to \mathrm{H}^{\cdot}(\overline{E}; \mathbb{Q})$$

is injective;

(b)

$$\mathrm{ch}(\overline{i}_! \mathcal{O}_X) = \overline{i}_!(1/ \operatorname{td} E)$$

in $\mathrm{H}^{\cdot}(\overline{E}; \mathbb{Q})$.

The result will follow since

$$\epsilon(\mathrm{ch}_X(\overline{i}_! \mathcal{O}_X)) = \mathrm{ch}(\overline{i}_! \mathcal{O}_X)$$

and

$$j^* \mathrm{ch}_X(\overline{i}_! \mathcal{O}_X) = \mathrm{ch}_X(i_! \mathcal{O}_X).$$

(a) Since

$$\mathrm{H}^{\cdot}_X(\overline{E}; \mathbb{Q}) \simeq \mathrm{H}^{\cdot}_X(E : \mathbb{Q}) \simeq \mathrm{H}^{\cdot}_{p-\mathrm{proper}}(E; \mathbb{Q})$$

the injectivity of ϵ will follow from the surjectivity of

$$\mathrm{H}^{\cdot}(\overline{E}; \mathbb{Q}) \to \mathrm{H}^{\cdot}(P(E); \mathbb{Q}).$$

Denote $\overline{U}$ (resp. U) the tautological line bundle on $\overline{E}$ (resp. $P(E)$). We know that $\mathrm{H}^{\cdot}(\overline{E}; \mathbb{Q})$ is a free $\mathrm{H}^{\cdot}(X; \mathbb{Q})$-module with basis

$$1, c_1(\overline{U}), \cdots, c_1(\overline{U})^r$$

and that $\mathrm{H}^{\cdot}(P(E);\mathbb{Q})$ is a free $\mathrm{H}^{\cdot}(X;\mathbb{Q})$-module with basis

$$1, c_1(U), \cdots, c_1(U)^{r-1}.$$

Since $\overline{U}_{|P(E)} \simeq U$, the conclusion follows.

(b) Since

$$\overline{U} \subset \overline{p}^{-1}(E \oplus (\mathbb{C} \times X))$$

we get a morphism

$$\overline{U} \to \overline{p}^{-1}E$$

and hence a section t of

$$F = \mathrm{Hom}\,(\overline{U}, \overline{p}^{-1}E) \simeq \overline{U}^* \otimes p^{-1}E.$$

One checks easily that t is transverse to the zero section of F and that $Z_t = X$. Therefore, Exercise 2.7.8 shows that

$$c_r(F) = \tau_{\overline{E}/X} = \overline{i}_!1.$$

Moreover, Lemma 5.7.2 shows that the Koszul complex

$$K.(F,t)$$

is a resolution of $\overline{i}_!\mathcal{O}_X$ by locally finite free $\mathcal{O}_E$-modules. It follows that

$$\mathrm{ch}(\overline{i}_!\mathcal{O}_X) = \mathrm{ch}\,K.(F,t) = \mathrm{ch}\bigwedge^{\cdot} F^*.$$

Using Proposition 4.4.9, we get

$$\mathrm{ch}(\overline{i}_!\mathcal{O}_X) = c_r(F)/\,\mathrm{td}\,F.$$

Note that

$$c_r(F)c_1(\overline{U}^*) = c_{r+1}(F \oplus \overline{U}^*) = 0$$

since

$$F \oplus \overline{U}^* \simeq \mathrm{Hom}\,(\overline{U}, \overline{p}^{-1}(E \oplus (\mathbb{C} \times X)))$$

has a nowhere vanishing section. But

$$\mathrm{td}(F)^{-1} = \mathrm{td}(\overline{U}^*)^{-1}\overline{p}^*\,\mathrm{td}(E)^{-1} \equiv \overline{p}^*\,\mathrm{td}(E)^{-1} \pmod{c_1(\overline{U}^*)}.$$

Therefore,

$$\begin{aligned} c_r(F)/\,\mathrm{td}\,F &= c_r(F)/\overline{p}^*\,\mathrm{td}\,E \\ &= (\overline{i}_!1)/\overline{p}^*\,\mathrm{td}\,E \\ &= \overline{i}_!(1/\,\mathrm{td}\,E). \end{aligned}$$

□

<u>Proof of Theorem 5.7.1</u>.

The idea is to use the deformation of X to the normal bundle of Y in order to reduce the general case to the case treated in Proposition 5.7.3.

Recall that with the complex deformation $\tilde{X}$ of X to the normal bundle $p : T_Y X \to Y$ is given a closed embedding $\tilde{i} : Y \times \mathbb{C} \to \tilde{X}$, a submersion $\tilde{\tau} : \tilde{X} \to \mathbb{C}$ and a projection $\tilde{p} : \tilde{X} \to X$ such that

(a) the maps
$$\tilde{p} \circ \tilde{i} : Y \times \mathbb{C} \to Y, \qquad \tilde{\tau} \circ \tilde{i} : Y \times \mathbb{C} \to \mathbb{C}$$
coincide with the canonical projections;

(b) there is a commutative diagram of the form
$$\begin{tikzcd} Y \times \{1\} \arrow[r, "\tilde{i}"] & \tilde{\tau}^{-1}(1) \\ Y \arrow[u, hook] \arrow[r, "i_1"'] & X \arrow[u, hook, "j_1"'] \end{tikzcd}$$

(c) there is a commutative diagram of the form
$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{i}"] & \tilde{\tau}^{-1}(0) \\ Y \arrow[u, hook] \arrow[r, "i_0"'] & T_Y X \arrow[u, hook, "j_0"'] \end{tikzcd}$$
with $\tilde{p} \circ j_0 = i_0 \circ p$.

Since both $\tilde{\tau}$ and $\tilde{\tau} \circ \tilde{i}$ are submersions, it follows that j_0 and j_1 are transverse to $\tilde{i}$. Hence,
$$Lj_0^* \tilde{i}_!(\mathcal{F} \boxtimes \mathcal{O}_{\mathbb{C}}) = i_{0!}(\mathcal{F})$$
and
$$Lj_1^* \tilde{i}_!(\mathcal{F} \boxtimes \mathcal{O}_{\mathbb{C}}) = i_{1!}(\mathcal{F}).$$
Setting $\mathcal{G} = \tilde{i}_!(\mathcal{F} \boxtimes \mathcal{O}_{\mathbb{C}})$, it follows that
$$\mathrm{ch}_Y(i_{0!}\mathcal{F}) = j_0^* \,\mathrm{ch}_{Y \times \mathbb{C}}(\mathcal{G})$$
and that
$$\mathrm{ch}_Y(i_{1!}\mathcal{F}) = j_1^* \,\mathrm{ch}_{Y \times \mathbb{C}}(\mathcal{G}).$$
Hence
$$j_{0!} \,\mathrm{ch}_Y(i_{0!}\mathcal{F}) = j_{0!}(1) \smile \mathrm{ch}_{Y \times \mathbb{C}}(\mathcal{G})$$
in $\mathrm{H}^{\cdot}_{Y \times \{0\}}(\tilde{X}; \mathbb{Q})$ and
$$j_{1!} \,\mathrm{ch}_Y(i_{1!}\mathcal{F}) = j_{1!}(1) \smile \mathrm{ch}_{Y \times \mathbb{C}}(\mathcal{G})$$

in $\mathrm{H}^{\cdot}_{Y\times\{1\}}(\tilde{X};\mathbb{Q})$. We have

$$j_{0!}(1) = \tau_{\tilde{X}/\tilde{\tau}^{-1}(0)} = \tilde{\tau}^*\tau_{\mathbb{C}/\{0\}}$$

in $\mathrm{H}^{\cdot}_{\tilde{\tau}^{-1}(0)}(\tilde{X};\mathbb{Z})$ and a similar formula for $j_{1!}(1)$. Since

$$\tau_{\mathbb{C}/\{0\}} = \tau_{\mathbb{C}/\{1\}}$$

in $\mathrm{H}^2_c(\mathbb{C};\mathbb{Z})$, they are equal in $\mathrm{H}^2_D(\mathbb{C};\mathbb{Z})$ for D a sufficient large compact disk. It follows that

$$j_{0!}(1) = j_{1!}(1)$$

in $\mathrm{H}^{\cdot}_{\tilde{\tau}^{-1}(D)}(\tilde{X};\mathbb{Z})$ and we get

$$\tilde{j}_0 \operatorname{ch}_Y(i_{0!}\mathcal{F}) = \tilde{j}_1 \operatorname{ch}_Y(i_{1!}\mathcal{F})$$

in $\mathrm{H}^{\cdot}_{Y\times D}(\tilde{X};\mathbb{Q})$. Since $Y \times D$ is a $\tilde{p}$-proper closed subset of $\tilde{X}$,

$$\tilde{p}_! : \mathrm{H}^{\cdot}_{Y\times D}(\tilde{X};\mathbb{Q}) \to \mathrm{H}^{\cdot}_Y(X;\mathbb{Q})$$

is well-defined and we get

$$\begin{aligned} \operatorname{ch}_Y(i_{1!}\mathcal{F}) &= \tilde{p}_! j_{1!} \operatorname{ch}_Y(i_{0!}\mathcal{F}) \\ &= \tilde{p}_! j_{0!} \operatorname{ch}_Y(i_{0!}\mathcal{F}). \end{aligned}$$

Using Proposition 5.7.3, we get finally

$$\begin{aligned} \operatorname{ch}_Y(i_{1!}\mathcal{F}) &= \tilde{p}_! j_{0!} i_{0!}(\operatorname{ch}\mathcal{F}/\operatorname{td} T_Y X) \\ &= i_{1!}\operatorname{ch}(\mathcal{F}/\operatorname{td} T_Y X) \end{aligned}$$

as requested.

5.8 Proof of Hirzebruch-Riemann-Roch theorem

Using the results in the three previous sections, we can now prove Theorem 5.1.10.

Proposition 5.8.1. *Assume $i : X \to Y$ is a closed embedding of compact complex analytic manifolds. Then Hirzebruch-Riemann-Roch theorem holds for X if it holds for Y.*

Proof. Let $\mathcal{F}$ be a coherent analytic sheaf on X. By Theorem 5.7.1, we have

$$\operatorname{ch}_X(i_!\mathcal{F}) = i_!(\operatorname{ch}\mathcal{F}/\operatorname{td} T_X Y).$$

Therefore,

$$\operatorname{ch}(i_!\mathcal{F}) \operatorname{td} TY = i_!(\operatorname{ch} \mathcal{F}\, i^* \operatorname{td} TY / \operatorname{td} T_X Y).$$

Thanks to the exact sequence

$$0 \to TX \to i^{-1}TY \to T_X Y \to 0$$

we have

$$i^* \operatorname{td} TY = \operatorname{td} TX \operatorname{td} T_X Y.$$

So

$$\operatorname{ch}_Y(i_!\mathcal{F}) \operatorname{td} TY = i_!(\operatorname{ch} \mathcal{F} \operatorname{td} TX)$$

and the conclusion follows. □

Corollary 5.8.2. *The Hirzebruch-Riemann-Roch theorem holds for any complex projective manifold.*

Proof. This follows directly from the preceding proposition if one takes $Y = \mathbb{P}_n(\mathbb{C})$ and uses Proposition 5.6.1. □

Bibliography

[1] P. Alexandroff and H. Hopf, *Topologie I*, Grundlehren der mathematischen Wissenschaften, no. 45, Springer, Berlin, 1935.

[2] M. F. Atiyah and F. Hirzebruch, *Riemann-Roch theorems for differentiable manifolds*, Bull. Amer. Math. Soc. **65** (1959), 276–281.

[3] M. F. Atiyah and F. Hirzebruch, *Analytic cycles on complex manifolds*, Topology **1** (1962), 25–45.

[4] M. F. Atiyah and F. Hirzebruch, *The Riemann-Roch theorem for analytic embeddings*, Topology **1** (1962), 151–166.

[5] A. Borel and J.-P. Serre, *Le théorème de Riemann-Roch*, Bull. Soc. Math. France **86** (1958), 97–136.

[6] G. E. Bredon, *Sheaf theory*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill, New York, 1967.

[7] J. Dieudonné, *Cours de géométrie algébrique I. Aperçu historique sur le développement de la géométrie algébrique.*, Le Mathématicien, no. 10, Presses Universitaires de France, Paris, 1974.

[8] ———, *A history of algebraic and differential topology. 1900–1960*, Birkhäuser, Boston, 1989.

[9] A. Dold, *Lectures on algebraic topology*, Grundlehren der mathematischen Wissenschaften, no. 200, Springer, Berlin, 1972.

[10] W.Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, no. 2, Springer, Berlin, 1984.

[11] R. Godement, *Topologie algébrique et théorie des faisceaux*, Hermann, Paris, 1964.

[12] H. Grauert and R. Remmert, *Theory of Stein spaces*, Grundlehren der mathematischen Wissenschaften, no. 236, Springer, Berlin, 1979.

[13] H. Grauert and R. Remmert, *Coherent analytic sheaves*, Grundlehren der mathematischen Wissenschaften, no. 265, Springer, Berlin, 1984.

[14] M. Greenberg, *Lectures on algebraic topology*, Mathematics Lecture Notes, Benjamin, New York, 1967.

[15] F. Hirzebruch, *Topological methods in algebraic geometry*, 3 ed., Grundlehren der mathematischen Wissenschaften, no. 131, Springer, Berlin, 1966.

[16] D. Husemoller, *Fibre bundles*, 2 ed., Graduate Texts in Mathematics, no. 20, Springer, New York, 1975.

[17] B. Iversen, *Local Chern classes*, Ann. scient. Éc. Norm. Sup. 4^e série **9** (1976), 155–169.

[18] M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Grundlehren der mathematischen Wissenschaften, no. 292, Springer, Berlin, 1990.

[19] S. Lefschetz, *Algebraic topology*, A. M. S. Colloqium Publications, no. XXVII, American Mathematical Society, New York, 1942.

[20] J. W. Milnor and J. D. Stasheff, *Characteritic classes*, Annals of Mathematics Studies, no. 76, Princeton University Press, Princeton, New Jersey, 1974.

[21] R. S. Palais and al, *Seminar on the Atiyah-Singer index theorem*, Annals of Mathematics Studies, no. 57, Princeton University Press, Princeton, 1965.

[22] P. Schapira and J.-P. Schneiders, *Elliptic pairs I: Relative finiteness and duality*, Astérisque **224** (1994), 5–60.

[23] ______, *Elliptic pairs II: Euler class and relative index theorem*, Astérisque **224** (1994), 61–98.

[24] J.-P. Schneiders, *Relative paracompactness as tautness condition in sheaf theory*, Bull. Soc. Roy. Sci. Liège **53** (1984), 179–186.

[25] H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, B. G. Teubner, Leipzig, 1934.

[26] J.-P. Serre, *Faisceaux algébriques cohérents*, Annals of Mathematics **61** (1955), 197–278.

[27] ———, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier (Grenoble) **6** (1956), 1–42.

[28] *Séminaire Heidelberg-Strasbourg 1966/67. Dualité de Poincaré*, Publications I.R.M.A. Strasbourg, no. 3, 1969.

[29] E. H. Spanier, *Algebraic topology*, McGaw-Hill Series in Higher Mathematics, McGraw-Hill, New York, 1966.

[30] N. Steenrod, *The topology of fiber bundles*, Princeton Mathematical Series, no. 14, Princeton University Press, Princeton, 1951.

[31] O. Veblen, *Analysis situs*, 2 ed., A. M. S. Colloqium Publications, no. V/II, American Mathematical Society, New York, 1931.

Montagem e impressão em :
SILVAS, crl • R. D. Pedro V, 126
Telef. 342 31 21 • 1250-094 LISBOA

Depósito Legal # 112 106/2000

TEXTOS DE MATEMÁTICA

Volumes já publicados

1 Luís Sanchez,
Métodos da Teoria de Pontos Críticos, 1993.

2 Miguel Ramos,
Teoremas de Enlace na Teoria dos Pontos Criticos, 1993.

3 Orlando Neto,
Aequaçõnes Diferenciais em Superfícies de Riemann, 1994.

4 A. Bivar Weinholtz,
Integral de Riemann e de Lebesgue em $\mathbb{R}^N$*, (*3^a *edição)*, 1999.

5 Mário S. R. Figueira,
Fundamentos de Análise Infinitesimal, (2^a *edição)*, 1997.

6 Owen J. Brison,
Teoria de Galois, (3^a *edição)*, 1999.

7 A. Bivar Weinholtz,
Aequaçõnes Diferenciais - Uma Introdução, (2^a *edição)*, 2000.

8 Armendo Machado,
Tópicos de Análise e Topologia em Variedades, 1997.

9 Armendo Machado,
Geometria Diferencial - Uma Introdução Fundamental, 1997.

10 A. Bivar Weinholtz,
Teoria dos Operadores, 1998.

11 Teresa Monteiro Fernandes,
Topologia Algébrica e Teoria elementar dos Feixes, 1998.

12 Owen J. Brison,
Grupos e Representações, 1999.

13 Jean-Pierre Schneiders,
Introduction to Characteristic Classes and Index Theory, 2000

Abstract

This book is based on a course given by the author at the university of Lisbon during the academic year 1997–1998. Its aim is to give the reader an idea of how the theory of characteristic classes can be applied to solve index problems. Starting from the Lefschetz fixed point theorem and its application to the computation of the Euler-Poincaré characteristic of a compact orientable manifold, we first develop the theory of Euler classes of orientable manifolds and real vector bundles. Then, we study Stiefel-Whitney classes and the general modulo 2 characteristic classes of real vector bundles. Similar considerations for complex vector bundles lead us to the Chern classes. We conclude the part devoted to characteristic classes by a study of global and local Chern characters. The rest of the book is then centered around the Riemann-Roch theorem. We present first a very simple proof which works for compact complex curves and allows us to make links with the original results of Riemann and Roch. Then, we treat in details the case of compact complex projective manifolds by more advanced methods.

Faculdade de Ciências — Departamento de Matemática
Rua Ernesto de Vasconcelos Bloco C1, 3° Piso
1749-016 LISBOA — PORTUGAL