A Coherence Criterion for Fréchet Modules

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1 Introduction

In the literature, one finds essentially two general criteria to get the finiteness of the cohomology groups of complexes of locally convex topological vector spaces. They are

(a) If $u^{\cdot} : G^{\cdot} \longrightarrow F^{\cdot}$ is a compact morphism of complexes of Fréchet spaces then $\dim H^k(F^{\cdot}) < +\infty$ for any $k \in \mathbb{Z}$ such that $H^k(u^{\cdot})$ is surjective.

(b) If $u^{\cdot} : G^{\cdot} \longrightarrow F^{\cdot}$ is a continuous morphism between a complex of (DFS) spaces and a complex of Fréchet spaces then $\dim H^k(F^{\cdot}) < +\infty$ for any $k \in \mathbb{Z}$ such that $H^k(u^{\cdot})$ is surjective.

The first of these criterion was used by Cartan-Serre [3] in order to get the finiteness of the cohomology groups of a compact complex analytic manifold with values in a coherent analytic sheaf. It was also used by Kashiwara [5] to get the constructibility of the solution complex associated to a holonomic differential module. The second criterion was proved by Bony-Schapira in [2]. Both are extensions of the Schwartz compact perturbation theorem.

In 1973, Houzel [4] has extended criterion (a) to complexes of modules over some sheaf of bornological algebras $\mathcal{A}$; the finiteness of the cohomology groups being replaced by the pseudo-coherence of the complex in the sense of SGA6 [1] (this is the only cohomological notion of finiteness which works well for a non necessarily coherent base algebra).

More precisely, Houzel assumes that $\mathcal{A}$ is a sheaf of bornological algebras which is complete and multiplicatively convex. The fibers of $\mathcal{A}$ are also assumed to be separated and to possess the homomorphism property. Then, working with complexes null in degree $\geq b$, he shows that in order to get the a-pseudocoherence of a complex $\mathcal{M}^{\cdot}$ of complete bornological $\mathcal{A}$-modules it is sufficient to find a sequence of $r \geq b - a + 1$ complexes of complete bornological $\mathcal{A}$-modules and bounded a-quasi-isomorphisms

$$\mathcal{M}_r \xrightarrow[u_r]{} \mathcal{M}_{r-1} \xrightarrow[u_{r-1}]{} \cdots \xrightarrow[u_1]{} \mathcal{M}$$

such that u_i is $\mathcal{A}$-nuclear in degree $\geq a$ and that the fibers of $\mathcal{M}_i^k$ are separated and possess the homomorphism property.

Using this theorem, Houzel shows that it is possible to give a simple proof of Grauert's coherence theorem. This theorem has also been used by Houzel and Schapira to give a criterion for the coherence of direct images of a coherent $\mathcal{D}$-module.

In [7], we found a situation where Houzel's result was not sufficient to solve the problem. What was needed was a generalization of the criterion (b) above. It is that extension which is the subject of this paper. It is contained in the following theorem.

Theorem 1.1 *Let X be a topological space with countable basis endowed with a multiplicatively convex sheaf of Fréchet algebras $\mathcal{A}$. Let $(\mathcal{M}_i)_{i\in\mathbb{N}}$, $\mathcal{M}$ be complexes of Fréchet $\mathcal{A}$-modules null in degree $\geq b$, let $u_{i+1,i} : \mathcal{M}_i \longrightarrow \mathcal{M}_{i+1}$ be $\mathcal{A}$-nuclear morphisms and let*

$$u : \varinjlim_{i\in\mathbb{N}} \mathcal{M}_i \longrightarrow \mathcal{M}$$

be a continuous morphism of $\mathcal{A}$-modules. Assume that u is an a-quasi-isomorphism and that any $x \in X$ has a fundamental system of open neighborhoods on which $\mathcal{M}$ and each $\mathcal{M}_i$ has enough sections. Then the complex $\mathcal{M}$ is a-pseudo-coherent over $\mathcal{A}$.

We refer the reader to §4 for definitions of the various concepts used in the previous statement. Note that the main difference with Houzel's results is that we only need *one* quasi-isomorphism to get the pseudo-coherence. This is important since, in practice, it is much more difficult to build quasi-isomorphisms than nuclear maps. For example, if X, S are complex analytic manifolds and $U \subset\subset V$ are two Stein open subsets of X then the restriction map

$$\pi_*(\mathcal{O}_{U\times S)} \longrightarrow \pi_*(\mathcal{O}_{V\times S})$$

is $\mathcal{O}_S$-nuclear.

The proof of this theorem will be found in §4. It has essentially the same structure as the proof of Houzel's finiteness theorem [4]. Despite our much weaker hypothesis, a proper use of Baire's theorem allows us to get the pseudo-coherence of the target complex.

2 Nuclear Perturbation Theorem

The basic finiteness tool used in this paper is Houzel's nuclear perturbation theorem. For the reader's convenience, we will give a quick proof of this result for Fréchet modules over a multiplicatively convex Fréchet algebra. We will avoid Houzel's bornological point of view since it is not important for our application (we are in a Fréchet framework). To fix the vocabulary, we first give some definitions.

Definition 2.1 A *Fréchet algebra* is a Fréchet space endowed with a continuous $\mathbb{C}$-bilinear multiplication

$$\cdot : A \times A \longrightarrow A$$

which is associative and admits a unit. We *do not* assume A commutative.

A Fréchet algebra is *multiplicatively convex* if any bounded subset B of A is absorbed by an absolutely convex subset of A stable for the multiplication law.

A (left) *Fréchet module* over a Fréchet algebra A is a Fréchet space M endowed with a structure of (left) A-module in such way that the action

$$\cdot : A \times M \longrightarrow M$$

is continuous.

A *morphism of Fréchet A-modules* is a continuous A-linear map. We denote by $L_A(E, F)$ the set of morphisms of the A-module E to the A-module F. This is naturally a $\mathbb{C}$-vector space. We turn it into a locally convex topological vector space by endowing it with the semi-norms

$$q_B(u) = \sup_{x \in B} q(u(x))$$

associated with the bounded subsets B of E and the semi-norms q of F. With this topology $L_A(E, F)$ is complete.

A morphism $u : E \longrightarrow F$ between two Fréchet A-modules is A-*nuclear* if

$$u(x) = \sum_{m=0}^{+\infty} \lambda_m \langle e^m, x \rangle f_m$$

where

- λ_m is a summable sequence in $\mathbb{C}$,
- e^m is a bounded sequence in $L_A(E, A)$,
- f_m is a bounded sequence in F.

We denote by $N_A(E, F)$ the $\mathbb{C}$-vector space of A-nuclear morphisms from E to F.

A morphism $u : E \longrightarrow F$ between two Fréchet A-modules is A-*finite* if

$$u(x) = \sum_{m=0}^{p} \langle e^m, x \rangle f_m$$

where

$$\begin{array}{rcl} e^0, \dots, e^p & \in & L_A(E, A) \\ f_0, \dots, f_p & \in & F \end{array}$$

In the rest of this section, A denotes a Fréchet algebra.

Proposition 2.2 *Let $u : E \longrightarrow F$, $v : F \longrightarrow G$ be two morphisms of Fréchet A-modules. Assume that either u or v is A-nuclear then so is $v \circ u$. Hence $N_A(E, F)$ is naturally a functor in E and F.*

Proof: Obvious. □

Proposition 2.3 [Lifting of nuclear morphisms]
Let $u : E \longrightarrow F$, $v : G \longrightarrow F$ be two morphisms of Fréchet A-modules. Assume u is surjective and v is A-nuclear. Then there is a morphism $w : G \longrightarrow E$ such that $u \circ w = v$.

Proof: Since v is A-nuclear one can find sequences $\lambda_m \in \mathbb{C}$, $e^m \in L_A(G,A)$, $f_m \in F$ such that $\sum_{m=0}^{\infty} |\lambda_m| < +\infty$, f_m being bounded in F and e^m being bounded in $L_A(G,A)$ in such a way that

$$v(x) = \sum_{m=0}^{+\infty} \lambda_m \langle e^m, x\rangle f_m.$$

Lemma 2.5 below shows that one can even assume f_m converges to 0 in F. Since u is surjective it is a strict morphism and one can find by Lemma 2.4 below a sequence $e_m \in E$ converging to 0 and such that $u(e_m) = f_m$. Let us define $w : G \longrightarrow E$ by setting

$$w(x) = \sum_{m=0}^{+\infty} \lambda_m \langle e^m, x\rangle e_m.$$

Obviously w is A-nuclear and $u \circ w = v$ as required. □

Lemma 2.4 *Let $u : E \longrightarrow F$ be a continuous $\mathbb{C}$ linear map between l.c.s. with a countable basis of semi-norms. Then u is a strict epimorphism if and only if for any sequence f_m of F converging to 0 there exist a sequence e_m of E converging to 0 such that $u(e_m) = f_m$.*

Proof: *The condition is necessary.*

Assume f_m converges to 0 in F. Let $(V_m)_{m\in\mathbb{N}}$ be a countable fundamental system of absolutely convex neighborhoods of 0 such that

$$m \in \mathbb{N} \Longrightarrow V_m \supset V_{m+1}$$

Since $u(V_m)$ is a neighborhood of 0 in F one can build a strictly increasing sequence of natural numbers M_k such that

$$m \geq M_k \Longrightarrow f_m \in u(V_k).$$

For $M_k \leq m < M_{k+1}$ let us choose an $e_m \in V_k$ such that $f_m = u(e_m)$. The sequence e_m converges to 0 in E as required.

The condition is sufficient.

Let V be a neighborhood of 0 in E. We need to show that $u(V)$ is a neighborhood of 0 in F. If it is not the case there is a sequence $f_m \in F\backslash u(V)$ which converges to 0 in F. Let e_m be a sequence in E converging to 0 and such that $u(e_m) = f_m$. There is an integer M such that $e_m \in V$ for $m \geq M$. For such an M, $f_M \in u(v)$ and this is impossible. □

Lemma 2.5 *Let λ_m be a sequence of complex numbers such that*

$$\sum_{m=0}^{\infty} |\lambda_m| < +\infty.$$

Then, there is a sequence r_m of positive real numbers converging to 0 such that

$$\sum_{m=0}^{+\infty} \frac{1}{r_m}|\lambda_m| < +\infty.$$

Proof: There is a strictly increasing sequence M_k of integers such that

$$\sum_{m=M_k}^{+\infty} |\lambda_m| < 2^{-2k}.$$

Let us define r_m to be 2^{-k} if $M_k \leq m < M_{k+1}$. We have

$$\sum_{m=M_k}^{M_{k+1}-1} \frac{1}{r_m}|\lambda_m| \leq 2^{-k}.$$

Hence,

$$\sum_{m=M_k}^{+\infty} \frac{1}{r_m}|\lambda_m| \leq 2^{-k} + 2^{-k-1} + \cdots \leq 2^{-k+1},$$

and the conclusion follows. □

Proposition 2.6 *Assume A is multiplicatively convex. Then, every A-nuclear endomorphism $u : E \longrightarrow E$ of the Fréchet A-module E may be written as*

$$u = u' + u''$$

where u' is A-finite and $1 - u''$ is invertible in $L_A(E, E)$.

Proof: Let

$$u(x) = \sum_{m=0}^{+\infty} \lambda_m \langle e^m, x\rangle f_m$$

with λ_m summable, e^m bounded in $L_A(E, A)$, f_m bounded in E. Since e^m is bounded in $L_A(E, A)$ and A is multiplicatively convex, there is a multiplicatively stable absolutely convex subset B_0 of A such that

$$\{\langle e^m, f_n\rangle : m, n \in \mathbb{N}\} \subset \mu B_0$$

where μ is a positive real number. Let us define u_p by setting

$$u_p = \sum_{m=p}^{+\infty} \lambda_m \langle e^m, x\rangle f_m.$$

We have:

$$u_p^k(x) = \sum_{m_k=p}^{+\infty} \cdots \sum_{m_1=p}^{+\infty} \lambda_{m_k} \cdots \lambda_{m_1} \langle e^{m_1}, x\rangle \langle e^{m_2}, f_{m_1}\rangle \cdots \langle e^{m_k}, f_{m_{k-1}}\rangle f_{m_k}.$$

Thus, for any semi-norm q in E and any bounded subset B of E, there is a constant C such that

$$\sup_{x\in B} q(u_p^k(x)) \leq C\Big(\sum_{m=p}^{+\infty} |\lambda_m|\Big)^k \mu^k$$

for any $p, k \in \mathbb{N}$. Choosing p such that $\mu \sum_{m=p}^{+\infty} |\lambda_m| = \varepsilon < 1$, it follows that

$$\sum_{k=0}^{+\infty} u_p^k$$

converges in $L_A(E, E)$ and $(1 - u_p) \sum_{k=0}^{+\infty} u_p^k = 1$. The conclusion follows since

$$u = \sum_{m=0}^{p-1} \lambda_m \langle e^m, x\rangle f_m + u_p.$$

□

Theorem 2.7 [Nuclear perturbation]

Let $u : E \longrightarrow F$, $v : E \longrightarrow F$ be two morphisms of Fréchet A-modules over a multiplicatively convex Fréchet algebra A. Assume u is surjective and v is A-nuclear. Then the A-module $\operatorname{coker}(u + v)$ *is finitely generated.*

Proof: Using Proposition 2.3, let us write $-v$ as $u \circ w$ where $w : E \longrightarrow E$ is an A-nuclear morphism. By the preceding proposition,

$$w = w' + w''$$

where w' is A-finite and $1 - w''$ is invertible in $L_A(E, E)$. We have

$$u + v = u - u \circ w = u \circ (1 - w) = u \circ (1 - w'') - u \circ w'.$$

Of course $u' = u \circ (1 - w'')$ is an epimorphism and $v' = u \circ w'$ is A-finite. Since they induce the same morphism from E to $\operatorname{coker}(u + v)$, $\operatorname{coker}(u + v)$ is finitely generated over A. □

3 Coherence over Fréchet Algebras

We will first consider the case where the base space X is reduced to a point and work with Fréchet modules over a multiplicatively convex Fréchet algebra.

Let us recall the notion of pseudo-coherence introduced in SGA6 [1].

Definition 3.1 Let A be any ring.

A morphism $u^\cdot : E^\cdot \longrightarrow F^\cdot$ of complexes of A-modules is an *a-quasi-isomorphism* if $H^a(u^\cdot)$ is an epimorphism and $H^k(u^\cdot)$ is an isomorphism for $k > a$. Equivalently, we can ask that $H^k(\operatorname{cone}(u)) = 0$ for $k \geq a$.

A complex of A-modules $E^\cdot$ is *perfect* if it is quasi-isomorphic to a bounded complex $F^\cdot$ such that F^k is a finite free A-module for every $k \in \mathbb{Z}$.

A complex of A-modules is *a-pseudo-coherent* if it is a-quasi-isomorphic to a perfect complex.

One checks that if in a distinguished triangle of complexes of A-modules,

$$E^\cdot \longrightarrow F^\cdot \longrightarrow G^\cdot \underset{+1}{\longrightarrow},$$

$E^\cdot$ and $G^\cdot$ are a-pseudo-coherent, then so is $F^\cdot$.

The global version of our coherence criterion is the following theorem.

Theorem 3.2 *Let A be a multiplicatively convex Fréchet algebra. Assume $(F_i^\cdot)_{i\in\mathbb{N}}$, $F^\cdot$ are complexes of Fréchet A-modules null in degree $\geq b$. Let $u_{i+1,i} : F_i^\cdot \longrightarrow F_{i+1}^\cdot$ be an A-nuclear morphism and let*

$$u : \varinjlim_{i\in\mathbb{N}} F_i^\cdot \longrightarrow F^\cdot$$

be a continuous morphism of A-modules. Assume u is an a-quasi-isomorphism. Then $F^\cdot$ is a-pseudo-coherent over A.

This result is a consequence of the two following lemmas.

Lemma 3.3 *Under the assumptions and notations of the theorem, let $c \in]a, b]$ and assume $H^k(F^\cdot) = 0$ for $k \geq c$. Then*

(i) For $i \gg 0$, there is an A-nuclear homotopy

$$h_i^\cdot : F_i^\cdot \longrightarrow F^\cdot[-1]$$

such that

$$v_i = f_i - d[-1] \circ h_i - h_i[1] \circ d$$

is zero in degrees $\geq c$.

(ii) $H^{c-1}(F^\cdot)$ is an A-module of finite type.

Proof: (i) We will proceed by decreasing induction on c (the case $c = b$ being clear).

For $i \gg 0$ there is an

$$h_{i+1}^\cdot : F_{i+1}^\cdot \longrightarrow F^\cdot[-1]$$

such that

$$v_{i+1} = f_{i+1} - d[-1] \circ h_{i+1}^\cdot - h_{i+1}[1] \circ d$$

is zero in degrees $\geq c + 1$.

We have

$$d^c \circ v_{i+1}^c = v_{i+1}^{c+1} \circ d_{i+1}^c = 0.$$

Thus, we get a morphism
$$v_{i+1}^c : F_{i+1}^c \longrightarrow Z^c.$$
The map
$$v_{i+1}^c \circ f_{i+1,i}^c : F_i^c \longrightarrow Z^c$$
is A-nuclear since $f_{i+1,i}^c$ is A-nuclear.

The differential
$$d^{c-1} : F^{c-1} \longrightarrow Z^c$$
is a continuous epimorphism since $H^c(F^\cdot) = 0$. Since both F^{c-1} and Z^{c-1} are Fréchet A-modules there is an A-nuclear morphism
$$h' : F_i^c \longrightarrow F^{c-1}$$
such that
$$d^{c-1} \circ h' = v_{i+1}^c \circ f_{i+1,i}^c.$$
Let us set
$$h'^k_i = h_{i+1}^k \circ f_{i+1,i}^k \quad \text{if} \quad k \neq c$$
and
$$h'^c_i = h_{i+1}^c \circ f_{i+1,i}^c + h'.$$
By construction the homotopy
$$h'_i : F_i^\cdot \longrightarrow F[-1]$$
is A-nuclear, and
$$v'_i = f_i - d[-1] \circ h'_i - h'_i[1] \circ d$$
is zero in degrees $\geq c$.

(ii) Let us take i great enough so that there is an homotopy
$$h_{i+1} : F_{i+1}^\cdot \longrightarrow F[-1]$$
which is A-nuclear and such that
$$v_{i+1} = f_{i+1} - d[-1] \circ h_{i+1} - h_{i+1}[1] \circ d$$
is zero in degrees $\geq c$. It is clear that v_{i+1}^{c-1} induces a morphism
$$v_{i+1}^{c-1} : F_{i+1}^{c-1} \longrightarrow Z^{c-1}.$$
The arrow
$$v_{i+1}^{c-1} \circ f_{i+1,i}^{c-1} : F_i^{c-1} \longrightarrow Z^{c-1}$$
is A-nuclear since $f_{i+1,i}^{c-1}$ is A-nuclear.

Let us denote by
$$u^i : F^{c-2} \oplus F_i^{c-1} \longrightarrow Z^{c-1}$$

the morphism defined by

$$u^i(a,b) = d^{c-2}(a) + v_{i+1}^{c-1} f_{i+1,i}^{c-1}(b).$$

Since $H^{c-1}(v_{i+1} \circ f_{i+1,i}) = H^{c-1}(f_i)$, the assumptions show that

$$\bigcup_{i \in \mathbb{N}} \text{im } u^i = Z^{c-1}.$$

Since Z^{c-1} and all the $F^{c-2} \oplus F_i^{c-1}$ are Fréchet A-modules, the conjunction of Baire's theorem and Banach's homomorphism theorem shows that

$$\operatorname{im} u^i = Z^{c-1}$$

for some $i \in \mathbb{N}$. The morphism u^i is thus an epimorphism, and, since $u^i(0,b)$ is A-nuclear, the nuclear perturbation theorem shows that $u^i(a,0)$ has a finitely generated cokernel. Hence $H^{c-1}(F^\cdot) = Z^{c-1}/\operatorname{im} d^{c-2}$ is finitely generated over A, and the proof is complete. □

Lemma 3.4 *Under the assumptions and the notations of the theorem, let $c \in [a,b]$ and assume $H^k(F^\cdot) = 0$ for $k \geq c$. Then $F^\cdot$ is a-pseudo-coherent over A.*

Proof: We proceed by increasing induction on c (the case $c = a$ being clear).

We know that $H^{c-1}(F^\cdot)$ is finitely generated over A. Hence, there is an epimorphism

$$u' : A^m \longrightarrow H^{c-1}(F^\cdot).$$

The morphism

$$\varinjlim_{i \in \mathbb{N}} H^{c-1}(F_i^\cdot) \longrightarrow H^{c-1}(F^\cdot)$$

is surjective. Hence, for $i \gg 0$, we can find morphisms

$$v_i : A^m \longrightarrow Z^{c-1}(F_i^\cdot)$$

which are compatible with the $f_{i+1,i}^{c-1}$ (i.e. $f_{i+1,i}^{c-1} \circ v_i = v_{i+1}$) and

$$v : A^m \longrightarrow Z^{c-1}(F^\cdot)$$

such that $f_i^{c-1} \circ v_i = v$. Moreover, we can ask that

$$p_{c-1} \circ v = u'$$

where $p_{c-1} : Z^{c-1}(F^\cdot) \longrightarrow H^{c-1}(F^\cdot)$ is the canonical projection.

This construction gives us an inductive system of morphisms of complexes of Fréchet A-modules

$$w_i : A^m[-(c-1)] \longrightarrow F_i^\cdot$$

and a morphism

$$w^\cdot : A^m[-(c-1)] \longrightarrow F^\cdot$$

such that $w^{\cdot} = f_i^{\cdot} \circ w_i^{\cdot}$ which induce an epimorphism

$$H^{c-1}(w^{\cdot}) : A^m \longrightarrow H^{c-1}(F^{\cdot}).$$

It is clear that the mapping cones of w and w_i are complexes of Fréchet A-modules null in degrees $\geq b$, the transition maps

$$\mathrm{cone}^{\cdot}(w_i) \longrightarrow \mathrm{cone}^{\cdot}(w_{i+1})$$

induced by $f_{i+1,i}$ being A-nuclear.

We get the following commutative diagram

$$\begin{array}{ccccccc}
A^m[-(c-1)] & \longrightarrow & \varinjlim_{i\in\mathbb{N}} F_i & \longrightarrow & \varinjlim_{i\in\mathbb{N}} \mathrm{cone}^{\cdot}(w_i) & \xrightarrow[+1]{} \\
\downarrow & & \downarrow & & \downarrow & \\
A^m[-(c-1)] & \longrightarrow & F^{\cdot} & \longrightarrow & \mathrm{cone}^{\cdot}(w) & \xrightarrow[+1]{}
\end{array}$$

where the lines are distinguished triangles. Since the first two vertical arrows are a-quasi-isomorphisms, so is the third one. From the second line we get the exact sequence

$$H^{c-1}(A^m[-(c-1)]) \xrightarrow[u^{\cdot}]{} H^{c-1}(F^{\cdot}) \longrightarrow H^{c-1}(\mathrm{cone}^{\cdot}(w)) \longrightarrow 0$$

and since u is surjective, $H^{c-1}(\mathrm{cone}^{\cdot}(w)) = 0$. Applying the induction hypothesis to $\mathrm{cone}^{\cdot}(w^{\cdot})$, we see that it is an a-pseudo-coherent complex. Since $A^m[-(c-1)]$ is perfect, the conclusion follows easily. □

4 Proof of Theorem 1.1

We will now extend the result established in the preceding section to the case of Fréchet modules over a sheaf of Fréchet algebras. Since the proof follows the same lines as in the absolute case, we shall give, in a few lemmas, the tools needed for the extension, and leave most of the obvious translation process to the reader.

Let X be a topological space with countable basis.

Recall that a *sheaf of Fréchet spaces* $\mathcal{F}$ on X is simply a sheaf with values in the category of Fréchet spaces and continuous linear maps. This means that for any countable covering $\mathcal{U}$ of an open subset U of X, $\Gamma(U;\mathcal{F})$ is the topological kernel of the usual Čech map

$$\prod_{V\in\mathcal{U}} \Gamma(V;\mathcal{F}) \longrightarrow \prod_{V,W\in\mathcal{U}} \Gamma(V\cap W;\mathcal{F}).$$

A *morphism of Fréchet sheaves* on X is a usual morphism of sheaves of $\mathbb{C}$-vector spaces which is continuous on the sections.

A *sheaf of Fréchet algebras* $\mathcal{A}$ on X is a sheaf of Fréchet spaces endowed with a $\mathbb{C}$-bilinear continuous multiplication

$$\cdot : \mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{A}$$

which is associative and admits a unit $1 \in \Gamma(X; \mathcal{A})$.

A sheaf of Fréchet algebras on X is *multiplicatively convex* if, for any open subset U of X, any bounded subset B of $\Gamma(U; \mathcal{A})$ and any $x \in X$, we can find a neighborhood V of x and a multiplicatively stable absolutely convex bounded subset B' of $\Gamma(V; \mathcal{A})$ absorbing $B_{|V}$.

A *Fréchet module* over a sheaf of Fréchet algebras $\mathcal{A}$ on X is a Fréchet sheaf $\mathcal{F}$ on X endowed with a structure of $\mathcal{A}$-module in such a way that the action map

$$\cdot : \mathcal{A} \times \mathcal{F} \longrightarrow \mathcal{F}$$

is continuous.

Let U be an open subset of X. A Fréchet module $\mathcal{F}$ over a sheaf of Fréchet algebras $\mathcal{A}$ on X *has enough sections on* U if for any $x \in U$ and any $f_x \in \mathcal{F}_x$ we may find a neighborhood V of x in X and

- λ_m : a summable sequence of complex numbers,
- f_m : a bounded sequence in $\Gamma(U; \mathcal{F})$,
- a_m : a bounded sequence in $\Gamma(V; \mathcal{A})$,

such that $f_x = g_x$ where $g \in \Gamma(V; \mathcal{F})$ is defined by

$$g = \sum_{m=0}^{\infty} \lambda_m a_m (f_m)_{|V}.$$

A *morphism of Fréchet modules* over a sheaf of Fréchet algebras $\mathcal{A}$ on X is a morphism of Fréchet sheaves which is $\mathcal{A}$-linear. We denote by $L_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ the set of morphisms from the Fréchet $\mathcal{A}$-module $\mathcal{E}$ to the Fréchet $\mathcal{A}$-module $\mathcal{F}$. It is naturally endowed with a structure of locally convex topological vector space with the semi-norms

$$q_B(u) = \sup_{e \in B} q(u_{|V}(e))$$

where V is an open subset of X, B a bounded subset of $\Gamma(V; \mathcal{E})$ and q a semi-norm of $\Gamma(V; \mathcal{F})$.

A morphism $u : \mathcal{E} \longrightarrow \mathcal{F}$ between two Fréchet $\mathcal{A}$-modules is $\mathcal{A}$-*nuclear* if for any $x \in X$ we can find an open neighborhood V of x and

- λ_m a summable sequence of complex numbers,
- e^m a bounded sequence of $L_{\mathcal{A}_{|V}}(\mathcal{E}_{|V}, \mathcal{A}_{|V})$,
- f_m a bounded sequence of $\Gamma(V; \mathcal{F})$,

such that

$$u(s) = \sum_{m=0}^{+\infty} \lambda_m e^m(s) f_{m|W}$$

for any open subset $W \subset V$ and any $s \in \Gamma(W, \mathcal{E})$. If in the preceding definition we use only finite sequences, we get the notion of $\mathcal{A}$-*finite* morphism.

Lemma 4.1 *Let $u : \mathcal{E} \longrightarrow \mathcal{F}$ be a morphism of Fréchet sheaves on X and let $x \in X$. Assume $u_x : \mathcal{E}_x \longrightarrow \mathcal{F}_x$ is surjective. Then any neighborhood U of x contains a neighborhood V such that*

$$\forall f \in \Gamma(U; \mathcal{F}) \quad \exists e \in \Gamma(V; \mathcal{E}) \qquad f_{|V} = u(e).$$

Proof: Of course, we may assume U is open. Let $(V_m)_{m\in\mathbb{N}}$ denote a countable fundamental system of open neighborhoods of x in U. For any $m \in \mathbb{N}$ consider the cartesian square

$$\begin{tikzcd} \Gamma(V_m; \mathcal{E}) \arrow[r, "u"] & \Gamma(V_m; \mathcal{F}) \\ G_m \arrow[u] \arrow[r, "u_m"'] \arrow[ur, phantom, "\square"] & \Gamma(U; \mathcal{F}). \arrow[u, "|V_m"'] \end{tikzcd}$$

By construction, G_m is a Fréchet space and the hypothesis implies that

$$\bigcup_{m\in\mathbb{N}} \operatorname{im} u_m = \Gamma(U; \mathcal{F}).$$

Hence, there is $m \in \mathbb{N}$ such that $\operatorname{im} u_m = \Gamma(U; \mathcal{F})$. This relation gives us the requested result. □

Using this lemma, the reader will easily check that in the sheaf version of Proposition 2.3, Proposition 2.6 and Theorem 2.7 the conclusions remain true locally.

Lemma 4.2 *Let $\mathcal{A}$ be a sheaf of Fréchet algebras on X, and let*

$$u_i : \mathcal{F}_i \longrightarrow \mathcal{F} \qquad (i \in \mathbb{N})$$

be a family of morphisms of Fréchet $\mathcal{A}$-modules. Assume that

$$\bigcup_{i\in\mathbb{N}} \operatorname{im} u_i = \mathcal{F}.$$

Assume moreover that $x \in X$ has a fundamental system of neighborhoods on which each $\mathcal{F}_i$ has enough sections. Then, for any $x \in X$, there is a neighborhood V of x and an integer $i \in \mathbb{N}$ such that

$$(u_i)_{|V} : (\mathcal{F}_i)_{|V} \longrightarrow \mathcal{F}_{|V}$$

is a sheaf epimorphism.

Proof: Denote by $\mathcal{V}$ a fundamental system of open neighborhoods of x on which each $\mathcal{F}_i$ has enough sections. Notice that $\mathcal{F}$ has also enough sections on any $V \in \mathcal{V}$.

Working as in the preceding lemma, it is easy to show that any $U \in \mathcal{V}$ contains a $V \in \mathcal{V}$ such that the map

$$\Gamma(V, \mathcal{F}_i) \times_{\Gamma(V,\mathcal{F})} \Gamma(U; \mathcal{F}) \longrightarrow \Gamma(U; \mathcal{F})$$

is surjective for some $i \in \mathbb{N}$. Let W be an open subset of V and assume

- λ_m is a summable sequence of complex numbers,
- a_m is a bounded sequence in $\Gamma(W;\mathcal{A})$,
- f_m is a bounded sequence in $\Gamma(U;\mathcal{F})$.

Then, we can find a bounded sequence $g_m \in \Gamma(V,\mathcal{F}_i)$ such that $(f_m)_{|V} = u_i(g_m)$. Hence,

$$\sum_{m=0}^{+\infty} \lambda_m a_m (f_m)_{|W} = u_i\left(\sum_{m=0}^{+\infty} \lambda_m a_m (g_m)_{|W}\right).$$

Combining this fact with the fact that $\mathcal{F}$ has enough sections on U, we see that

$$(u_i)_{|W} : (\mathcal{F}_i)_{|V} \longrightarrow \mathcal{F}_{|V}$$

is a sheaf epimorphism. □

With the preceding lemmas at hand, we can prove Theorem 1.1 by working as in the proof of Theorem 3.2 but in the context of sheaves. For the sake of brevity we leave this straightforward rewriting to the reader.

5 An application to analytic geometry

In this section, we will give an example of application of our finiteness criterion in the case of topological modules over the algebra $\mathcal{O}_S$ of holomorphic functions on a complex manifold S. This corollary is used in [7] to get the relative finiteness theorem for elliptic pairs.

Let S be a complex analytic manifold. Recall that the sheaf $\mathcal{O}_S$ of holomorphic functions on S is a multiplicatively convex sheaf of Fréchet algebras over S (see [4]). Also recall that if V is a relatively compact open subset of a Stein open subset U of X, then the restriction map

$$\Gamma(U;\mathcal{O}_S) \longrightarrow \Gamma(V;\mathcal{O}_S)$$

is $\mathbb{C}$-nuclear. From this it follows easily that $\Gamma(U;\mathcal{O}_U)$ is a Fréchet nuclear (FN) space and that $\Gamma(\overline{V},\mathcal{O}_S)$ is a dual Fréchet nuclear (DFN) space.

Following [6], an FN-free (resp. a DFN-free) $\mathcal{O}_S$-module is a module isomorphic to $E \mathbin{\hat{\otimes}} \mathcal{O}_S$ for some Fréchet nuclear (resp. dual Fréchet nuclear) space E.

Corollary 5.1 *Let $\mathcal{M}^\cdot$ (resp. $\mathcal{N}^\cdot$) be a complex of DFN-free (resp. FN-free) $\mathcal{O}_S$-modules. Assume $\mathcal{M}^\cdot$ and $\mathcal{N}^\cdot$ are bounded from above and*

$$u^\cdot : \mathcal{M}^\cdot \longrightarrow \mathcal{N}^\cdot$$

is a continuous $\mathcal{O}_S$-linear quasi-isomorphism. Then $\mathcal{M}^\cdot$ and $\mathcal{N}^\cdot$ have $\mathcal{O}_S$-coherent cohomology.

Proof: Let E be a (DFN)-space and set $\mathcal{E} = E \hat{\otimes} \mathcal{O}_S$. It is well-known that we can find a countable inductive system $(F_n, f_{mn})_{n\in\mathbb{N}}$ of Fréchet spaces with nuclear transition maps such that

$$\varinjlim_{n\in\mathbb{N}} F_n \xrightarrow{\sim} E.$$

Let us denote by $f_n : F_n \longrightarrow E$ the projection to the limit. Since E is separated, $\ker f_n$ is a Fréchet subspace of F_n, and it follows from the equality

$$\ker f_n = \bigcup_{m\geq n} \ker f_{mn}$$

that $\ker f_n = \ker f_{mn}$ for $m \gg 0$. The sheaf $\mathcal{F}_n = F_n \hat{\otimes} \mathcal{O}_S$ is obviously a sheaf of Fréchet modules over the Fréchet algebra $\mathcal{O}_S$ and each transition map $\phi_{mn} = f_{mn} \hat{\otimes} \mathrm{id}_{\mathcal{O}_S}$ is clearly $\mathcal{O}_S$-nuclear. Using the maps $\phi_n = f_n \hat{\otimes} \mathrm{id}_{\mathcal{O}_S}$, we get the isomorphism

$$\varinjlim_{n\in\mathbb{N}} \mathcal{F}_n \xrightarrow{\sim} \mathcal{E}.$$

Moreover, locally on S,

$$\ker \phi_n = \ker \phi_{mn} \quad \text{for} \quad m \gg 0. \tag{5.1}$$

Now, let E^0, E^1 be two (DFN) spaces and consider a continuous $\mathcal{O}_S$-linear morphism

$$u : \mathcal{E}^0 \longrightarrow \mathcal{E}^1$$

between the associated DFN-free $\mathcal{O}_S$-modules. As above,

$$\mathcal{E}^0 = \varinjlim_{n\in\mathbb{N}} \mathcal{F}^0_n, \quad \mathcal{E}^1 = \varinjlim_{n\in\mathbb{N}} \mathcal{F}^1_n,$$

where $(\mathcal{F}^0_n, \phi^0_{mn})$, $(\mathcal{F}^1_n, \phi^1_{mn})$ are inductive systems of FN-free $\mathcal{O}_S$-modules with $\mathcal{O}_S$-nuclear transition maps. Let us fix $n \in \mathbb{N}$. Working as in Lemma 4.1 it is easy to see that for $m \gg 0$, there is a map $u_n : \mathcal{F}^0_n \longrightarrow \mathcal{F}^1_m$ such that $u \circ \phi_n = \phi_m \circ u_n$. Hence, locally, thanks to (5.1), it is possible to find a strictly increasing sequence $k_n \in \mathbb{N}$ and a morphism

$$(u_n) : (\mathcal{F}^0_n) \longrightarrow (\mathcal{F}^1_{k_n})$$

of inductive systems such that $\varinjlim_{m\in\mathbb{N}} u_n = u$.

Finally, assume we have a complex $\mathcal{E}^{\cdot}$ of DFN-free $\mathcal{O}_S$-modules. Using the preceding procedure and (5.1) one sees that it is possible to find an inductive system of complexes of FN-free $\mathcal{O}_S$-modules $(\mathcal{F}^{\cdot}_n, \phi^{\cdot}_{mn})_{n\in\mathbb{N}}$ with $\mathcal{O}_S$-nuclear transition maps such that

$$\varinjlim_{n\in\mathbb{N}} \mathcal{F}^{\cdot}_n \simeq \mathcal{E}^{\cdot}.$$

The conclusion then follows easily from Theorem 1.1 by using the well-known fact that FN-free $\mathcal{O}_S$-modules have enough sections on polydiscs. $\square$

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